Zbierka riešených príkladov z kombinatorickej analýzy

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Obsah

1 Úvod

Štúdium informatiky si vyžaduje istú zbehlosť pri riešení enumeračných a kombinatorických úloh, schopnosť riešiť rekurentné vzťahy a konštruovať asymptotické odhady. Kombinatorika sa na FMFI UK prednáša v rozličných kurzoch; na informatike to je minimálne diskrétna matematika I (v súčasnosti Úvod do kombinatoriky a teórie grafov), kombinatorická analýza I a II. Skúsenosti zo skúšok z kombinatorickej analýzy ukázali, že študenti pomerne dobre chápu teóriu, ale chýbajú im praktické skúsenosti pri riešení kombinatorických úloh. Zložitosť týchto úloh, veľký počet študentov a slabá časová dotácia cvičení neumožňuje získať potrebné skúsenosti na cvičeniach, ale vyžaduje si samostatnú prácu. To však naráža na nedostatok primerane náročných kombinatorických úloh. Tento dokument predstavuje zberku riešených príkladov z kombinatorickej analýzy. Vznikla čiastočne na základe rozličných literálnych zdrojov a čiastočne z príkladov, ktoré sme vymýšľali na skúšky. Tématicky je členená podobne ako Knuthova Concrete mathematics [1]; na riešenie rekurentných vzťahov, súm, súm s celými časťami, kombinatorických vzťahov, výpočty súm obsahujúcich harmonické čísla, Fibonacciho čísla, na úlohy na generujúce funkcie a na výpočty asymptotických odhadov. Tento dokument predstavuje pracovnú verziu, ktorá je neustále dopĺňaná novými prikladmi. Preto nie je organizovaná ako tradičné zbierky (teória, vzorové riešenie, priklady, riešenia). Časom ju možno do tejto podoby upravíme. Čitateľovi odporúčame, aby sa najprv pokúsil riešiť zadané úlohy a až potom sa pozrel na riešenia.

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Autori

2 Elementárne konečné sumy

$\boxed{\text{SE1}}$

$$\sum_0^a x \cdot 2^x \delta x = x \cdot 2^x|_0^a - \sum_0^a 2^{x+1} \delta x = a \cdot 2^a - 2^{a+1} + 2$$

$\boxed{\text{SE2}}$

$$\sum_0^n x^{\underline{m}} \delta x = \frac{x^{\underline{m+1}}}{m+1}|_0^n = \frac{n^{\underline{m+1}}}{m+1} \qquad m \neq -1.$$

$\boxed{\text{SE3}}$

$$\sum_0^n x^{\underline{-1}} \delta x = H_n.$$

$\boxed{\text{SE4}}$

$$\sum_0^{n-1} k^2 = \sum_0^n x^2 \delta x = \sum_0^n x^{\underline{2}} + x^{\underline{1}} \delta x = \frac{n^{\underline{3}}}{3} + \frac{n^{\underline{2}}}{2} = \frac{n(n-1)(2n-1)}{6}$$

$\boxed{\text{SE5}}$

$$\sum_{k=0}^{2m} (-1)^k k^3 = \sum_{k=1}^{m} (2k)^3 - (2k-1)^3 = 12 \sum_{k=1}^{m} k^2 - 6 \sum_{k=1}^{m} k + \sum_{k=1}^{m} 1 = m^2(4m+3).$$

$\boxed{\text{SE6}}$

$$\sum_{k=0}^{n} (-1)^k k^3 = \begin{cases} m^2(4m+3) & n = 2m \\ \\ m^2(4m+3) - (2m+1)^3 & n = 2m+1 \end{cases}$$

$\boxed{\text{SE7}}$

$$\begin{aligned} S_n &= \sum_{k=0}^{n} \frac{(-1)^k k}{4k^2-1} = \sum_{k=1}^{n} \frac{(-1)^k k}{(2k+1)(2k-1)} = \frac{1}{4}\left(\sum_{k=1}^{n} \frac{(-1)^k}{(2k+1)} + \sum_{k=1}^{n} \frac{(-1)^k}{(2k-1)} \right) = \\ &= \frac{1}{4}\left(-\sum_{k=2}^{n+1} \frac{(-1)^k}{(2k-1)} + \sum_{k=1}^{n} \frac{(-1)^k}{(2k-1)} \right) = \frac{1}{4}\left(\frac{(-1)^n}{2n+1} - 1 \right). \end{aligned}$$

3 Sumy s harmonickými číslami

$\boxed{\text{SH1}}$

$$\sum_{k=0}^{n} \frac{(-1)^k}{(k+1)} = \sum_{k=1}^{n+1} \frac{1}{k} - 2 \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{1}{2k} = H_{n+1} - H_{\lfloor n/2 \rfloor}.$$

$\boxed{\text{SH2}}$

$$\sum_{k=0}^{n} H_k = \sum_{0}^{n} H_x \delta x + H_n = xH_x|_0^n - \sum_{0}^{n} (x+1)^{\underline{1}} \frac{1}{x+1} \delta x + H_n = nH_n - n + H_n = (n+1)H_n - n.$$

$\boxed{\text{SH3}}$

$$\begin{aligned} S_{2m} &= \sum_{k=0}^{2m} (-1)^k \cdot H_k = H_0 + (H_2 - H_1) + (H_4 - H_3) + \cdots + (H_{2m} - H_{2m-1}) = \\ &= 0 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2m} = \frac{1}{2} H_m. \end{aligned}$$

$\boxed{\text{SH4}}$

$$\sum_{k=1}^{n} H_{2k-1} = \frac{1}{2} \sum_{k=1}^{2n} \left[H_k - (-1)^k H_k \right] = \frac{(2n+1)H_{2n} - (2n) - \frac{1}{2} H_n}{2}.$$

$\boxed{\text{SH5}}$

$$\begin{aligned} S_{2m} &= \sum_{k=0}^{2m} (-1)^k k H_k = \sum_{k=1}^{m} [(2k)H_{2k} - (2k-1)H_{2k-1}] = \sum_{k=1}^{n} H_{2k+1} + m = \\ &= \frac{1}{2} \left[(2m+1)H_{2m} - \frac{1}{2} H_m - 2m \right] + m = \frac{1}{2} \left[(2m+1)H_{2m} - \frac{1}{2} H_m \right]. \end{aligned}$$

$\boxed{\text{SH6}}$

$$\sum_{0}^{n} x^2 H_x \delta x = \frac{x^{\underline{3}}}{3} H_x|_0^n - \frac{1}{3} \sum_{0}^{n} \frac{(x+1)^{\underline{3}}}{x+1} \delta x = \frac{n^{\underline{3}}}{3} H_n - \frac{1}{3} \sum_{0}^{n} x^{\underline{2}} \delta x = \frac{n^{\underline{3}}}{3} H_n - \frac{n^{\underline{3}}}{9}$$

$\boxed{\text{SH7}}$

$$\begin{aligned} S_n &= \sum_{0}^{n} x^{\underline{m}} H_x \delta x = \frac{x^{\underline{m+1}}}{m+1} H_x|_0^n - \frac{1}{m+1} \sum_{0}^{n} \frac{(x+1)^{\underline{m+1}}}{x+1} \delta x = \frac{n^{\underline{m+1}}}{m+1} H_n - \frac{1}{m+1} \sum_{0}^{n} x^{\underline{m}} \delta x = \\ &= \frac{n^{\underline{m+1}}}{m+1} H_n - \frac{n^{\underline{m+1}}}{(m+1)^2}. \end{aligned}$$

$\boxed{\text{SH8}}$

$$\begin{aligned} S_n &= \sum_{k=0}^{n} k^2 H_k = \sum_{0}^{n} x^2 H_x \delta x + n^2 H_n = \sum_{0}^{n} x^{\underline{2}} H_x \delta x + \sum_{0}^{n} x^{\underline{1}} H_x \delta x + n^2 H_n = \\ &= \frac{3n^{\underline{3}} H_n - n^{\underline{3}}}{9} + \frac{2n^{\underline{2}} H_n - n^{\underline{2}}}{4} + n^2 H_n. \end{aligned}$$

4 Sumy s celými časťami

$\boxed{\text{SI1}}$

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \lfloor \log_2 k \rfloor) = \sum_{k,m} (\lfloor \log_2 k \rfloor = m) \cdot m \cdot (1 \le k \le n) = \sum_{k,m} (m \le \log_2 k < m+1)(1 \le k \le n) = \\ &= \sum_{m,k} (2^m \le k < 2^{m+1}) \cdot m \cdot (1 \le m \le a) + a \cdot (n - 2^a + 1) = \sum_m^a m \cdot 2^m + a \cdot (n - 2^a + 1) = \\ &= (n+1) \cdot a - 2^{(a+1)} + 2; \qquad a = \lfloor \log_2 n \rfloor. \end{aligned}$$

$\boxed{\text{SI2}}$

$$S_n = \sum_{k=1}^{n} \lfloor \log_2 (2k) \rfloor = \sum_{k=1}^{n} (1 + \lfloor \log_2 k \rfloor) = (n+1) \cdot a - 2^{(a+1)} + 2 + n; \qquad a = \lfloor \log_2 n \rfloor$$

$\boxed{\text{SI3}}$

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \lfloor \log_2 (2k+1) \rfloor \quad (\lfloor \log_2 (2k+1) \rfloor = \lfloor \log_2 (2k) \rfloor; \quad k > 0) \\ S_n &= \sum_{k=1}^{n} \lfloor \log_2 (2k) \rfloor = (n+1) \cdot a - 2^{(a+1)} + 2 + n; \qquad a = \lfloor \log_2 n \rfloor \end{aligned}$$

$\boxed{\text{SI4}}$

$$\begin{aligned} \sum_{k=0}^{n} \lfloor \log_2 (2k+3) \rfloor &= \sum_{k=1}^{n+1} \lfloor \log_2 (2k+1) \rfloor = \lfloor \log_2 (2n+3) \rfloor + \sum_{k=1}^{n} \lfloor \log_2 (2k+1) \rfloor = \\ &= \lfloor \log_2 (2n+3) \rfloor + (n+1) \cdot a - 2^{(a+1)} + 2 + n; \qquad a = \lfloor \log_2 n \rfloor \end{aligned}$$

$\boxed{\text{SI5}}$

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \lfloor \log_2 (k+1) \rfloor - \lceil \log_2 k \rceil = \lfloor \log_2 (n+1) \rfloor + \sum_{k=1}^{n} \lfloor \log_2 k \rfloor - \lceil \log_2 k \rceil = \\ &= \lfloor \log_2 (n+1) \rfloor + \sum_{k,j} ((k = 2^j) - 1)(1 \le k \le n) = \lfloor \log_2 (n+1) \rfloor - n + \\ &+ \sum_{k,j} (k = 2^j)(1 \le k \le n) = \lfloor \log_2 (n+1) \rfloor - n + \sum_j (0 \le j \le \lfloor \log_2 n \rfloor) = \\ &= \lfloor \log_2 (n+1) \rfloor - n + \lfloor \log_2 n \rfloor + 1 \end{aligned}$$

$\boxed{\text{SI6}}$

$$\begin{aligned} S_{2m} &= \sum_{k=1}^{2m} (-1)^k \lfloor \log_2 k \rfloor = \sum_{k=1}^{m} \lfloor \log_2 2k \rfloor - \sum_{k=1}^{m-1} \lfloor \log_2 (2k+1) \rfloor = \\ &= \sum_{k=1}^{m} \lfloor \log_2 2k \rfloor - \sum_{k=1}^{m-1} \lfloor \log_2 2k \rfloor = \lfloor \log_2 2m \rfloor; \qquad \lfloor \log_2 (2k+1) \rfloor = \lfloor \log_2 2k \rfloor; \quad k \ge 1. \end{aligned}$$

$\boxed{\text{SI7}}$

$$\sum_{k=1}^{n}(-1)^k\lfloor\log_2 k\rfloor = \begin{cases} \lfloor\log_2 n\rfloor & n \text{ je párne} \\ 0 & n \text{ je nepárne} \end{cases}$$

$\boxed{\text{SI8}}$

$$\sum_{k=1}^{n}(-1)^k\lfloor\log_2 2k\rfloor = \sum_{k=1}^{n}(-1)^k\lfloor\log_2 k\rfloor + (-1)^k = \begin{cases} \lfloor\log_2 n\rfloor & n \text{ je párne} \\ -1 & n \text{ je nepárne} \end{cases}$$

$\boxed{\text{SI9}}$

$$\begin{aligned} \sum_{k\geq 1}\binom{n}{\lfloor\log_m k\rfloor} &= \sum_{k,r}\binom{n}{r}\cdot(r=\lfloor\log_m k\rfloor) = \sum_{k,r}\binom{n}{r}\cdot(r\leq\log_m k<r+1) = \\ &= \sum_{k,r}\binom{n}{r}\cdot(m^r\leq k<m^{r+1}) = \sum_{k,r}\binom{n}{r}\cdot m^r(m-1) = \\ &= (m-1)\sum_{k,r}\binom{n}{r}\cdot m^r = (m-1)(m+1)^n. \end{aligned}$$

$\boxed{\text{SI10}}$

$$\begin{aligned} S_n &= \sum_{k=1}^{n}\lfloor\sqrt{k^2+1}\rfloor - \lceil\sqrt{k^2-1)}\rceil, \qquad n\geq 1. \\ & k^2 < k^2+1 < (k+1)^2; \quad k>0; \\ & k < \sqrt{k^2+1} < (k+1) \\ & k = \lfloor\sqrt{k^2}\rfloor \leq \lfloor\sqrt{k^2+1}\rfloor \leq \sqrt{k^2+1} < (k+1); \\ & \lfloor\sqrt{k^2+1}\rfloor = k, \quad k>0 \\ & \lceil\sqrt{k^2-1)}\rceil = 0, \quad k=1; \\ & (k-1)^2 < k^2-1 < k^2; \quad k>1; \\ & (k-1) < \sqrt{k^2-1} < k; \\ & (k-1) < \lceil\sqrt{k^2-1)}\rceil \leq \lceil\sqrt{k^2}\rceil = k; \\ & \lceil\sqrt{k^2-1)}\rceil = k. \\ S_n &= 1+\sum_{k=2}^{n}\lfloor\sqrt{k^2+1}\rfloor - \lceil\sqrt{k^2-1)}\rceil = 1+\sum_{k=2}^{n}(k-k) = 1. \end{aligned}$$

5 Fibonacciho čísla

$$F_n = F_{n-1}+F_{n-2}; \quad F_0 = 0, F_1 = 1; \qquad \Delta F_k = F_{k-1}; \quad \Delta F_{2k} = F_{2k-1}, \quad \Delta F_{2k+1} = F_{2k}$$

$$F(z) = \frac{z}{1-z-z^2}$$

k	0	1	2	3	4	5	6	7	8	9	10
F_k	0	1	1	2	3	5	8	13	21	34	55

F1

$$\sum_{k=0}^{n} F_k = \sum_{0}^{n+1} F_x \delta x = F_{x+1}|_0^{n+1} = F_{n+2} - F_1 = F_{n+2} - 1$$

F2

$$\sum_{k=0}^{n} F_{2k} = \sum_{k=0}^{n} F_{2x+2} \delta x = F_{2x+1}|_0^{n} = F_{2n+1} - 1.$$

F3

$$\sum_{k=0}^{n} F_{2k+1} = \sum_{k=0}^{n+1} F_{2x+1} \delta x = F_{2x}|_0^{n+1} = F_{2n+2}$$

F4

$$\sum_{k=0}^{2m} (-1)^k F_k = \sum_{k=0}^{m} F_{2k} - \sum_{k=0}^{m-1} F_{2k+1} = F_{2m-1} + 1$$

F5

$$\sum_{k=0}^{n} kF_k = \sum_{0}^{n+1} xF_x \delta x = xF_{x+1}|_0^{n+1} - \sum_{0}^{n+1} F_{x+2} \delta x = (n+1)F_{n+2} - F_{n+4} + F_3 = nF_{n+2} - F_{n+3} + 2$$

F6

$$\sum_{k=0}^{n} (n-k)F_k = n\sum_{k=0}^{n} F_k - \sum_{k=0}^{n} kF_k = F_{n+3} - n - 2$$

F7

$$\sum_{k=0}^{n} (k)F_{n-k} = [z^n]\frac{z}{(1-z)^2}\frac{z}{1-z-z^2} = [z^n]\frac{-1}{(1-z)^2} + \frac{-1}{(1-z)} + \frac{z+2}{(1-z-z^2)}$$

$$= -(n+1) - 1 + F_n + 2F_{n+1} = F_{n+3} - n - 2$$

F8

$$S_m \quad = \quad 2\sum_{k=0}^{m-1} kF_{2k} = 2\sum_{0}^{m} xF_{2x} \delta x = 2xF_{2x-1}|_0^{m} - 2\sum_{0}^{m} F_{2x+2} \delta x = 2mF_{2m-1} - 2(F_{2m} - 1)$$

$\boxed{\text{F9}}$

$$\begin{aligned} S_{2m} &= \sum_{k=0}^{2m}(-1)^k kF_k = \sum_{k=1}^{m}(2kF_{2k}-(2k-1)F_{2k-1}) = \sum_{k=1}^{m}2k(F_{2k}-F_{2k-1}) + \sum_{k=0}^{m-1}F_{2k+1} = \\ &= 2\sum_{k=1}^{m}kF_{2k-2} + F_{2m} - 1 = F_{2m} - 1 + 2\sum_{k=0}^{m-1}(k+1)F_{2k} = F_{2m} - 1 + 2\sum_{k=0}^{m-1}kF_{2k} + \\ &+ 2\sum_{k=0}^{m-1}F_{2k} = (2m+2)F_{2m-1} - F_{2m} + 1 \end{aligned}$$

$\boxed{\text{F10}}$

$$\begin{aligned} S_n &= \sum_{k=0}^{n}k^2\cdot F_{n-k} = [z^n](zD)^2(\frac{1}{1-z})\cdot F(z) = \frac{z+z^2}{(1-z)^3}\cdot\frac{z}{1-z-z^2} = \\ &= [z^n]\left(\frac{-2}{(1-z)^3} - \frac{1}{(1-z)^2} - \frac{5}{1-z} + \frac{8+5z}{(1-z-z^2)}\right) = \\ &= 8F_{n+1} + 5F_n - (n^2+4n+8). \end{aligned}$$

$\boxed{\text{F11}}$

$$\begin{aligned} S_n &= \sum_{k=0}^{n}k^2\cdot F_k = \sum_{k=0}^{n}[(n-k)^2 - n^2 + 2nk]\cdot F_k = \sum_{k=0}^{n}(n-k)^2]\cdot F_k - n^2\cdot\sum_{k=0}^{n}F_k + \\ &+ 2n\sum_{k=0}^{n}k\cdot F_k \end{aligned}$$

$\boxed{\text{F12}}$

$$\begin{aligned} S_n &= \sum_{k=0}^{n}k^2F_k = \sum_{0}^{n+1}x^{\underline{2}}F_x\delta x = x^{\underline{2}}F_{x+1}|_0^{n+1} - 2\sum_{0}^{n+1}xF_{x+2}\delta x = (n^2+n)F_{n+2} \\ &- 2\cdot\left[xF_{x+3}|_0^{n+1} - \sum_{0}^{n+1}F_{x+4}\delta x\right] = (n^2+n)F_{n+2} - 2\cdot[(n+1)F_{n+4} - F_{n+6} + F_5] = \\ &= (n^2+n)F_{n+2} - (2n+2)F_{n+4} + 2F_{n+6} - 10. \end{aligned}$$

$\boxed{\text{F13}}$

$$\begin{aligned} S_n &= \sum_{k=0}^{n}k^2F_{k+1} = \sum_{k=1}^{n+1}(k-1)^2F_k = n^2F_{n+1} + \sum_{k=1}^{n}(k^2-2k+1)F_k = n^2F_{n+1} + \sum_{k=1}^{n}(k^2-k)F_k - \\ &- \sum_{k=1}^{n}kF_k + \sum_{k=1}^{n}F_k = n^2F_{n+1} + (n^2+n)F_{n+2} - (2n+2)F_{n+4} + 2F_{n+6} - 10 - nF_{n+2} + \\ &+ F_{n+3} - 2 + F_{n+2} - 1 = n^2F_{n+3} - (2n+1)F_{n+4} + 2F_{n+6} - 13. \end{aligned}$$

$\boxed{\text{F14}}$ Nájdite vytvárajúcu funkciu pre postupnosť $a_n = \sum_{k=0}^{n} F_{2k}$.

$$\begin{aligned}
\{F_{2k}\} &\leftrightarrow \frac{z}{1-3z+z^2} \\
\left\{\sum_{k=0}^{n} F_{2k}\right\} &\leftrightarrow \frac{z}{1-3z+z} \cdot \frac{1}{1-z}.
\end{aligned}$$

$\boxed{\text{F15}}$ Spočítajte sumu

$$\begin{aligned}
S_n &= \sum_{0\le j\le k\le n} k\cdot F_j = \sum_{0\le k\le n} k\cdot \sum_{0\le j\le k} F_j = \sum_{0\le k\le n} k\cdot(F_{k+2}-1) = \sum_{0\le k\le n} k\cdot F_{k+2} - \sum_{0\le k\le n} k = \\
&= \sum_{0\le k-2\le n} (k-2)\cdot F_k - \binom{n+1}{2} = \sum_{2\le k\le n+2} k\cdot F_k - 2\sum_{2\le k\le n+2} F_k - \binom{n+1}{2} = \\
&= \sum_{0\le k\le n+2} k\cdot F_k - 1\cdot F_1 - 2\sum_{0\le k\le n+2} F_k + 2F_1 - \binom{n+1}{2} = (n+2)F_{n+4} - F_{n+5} + 2 - \\
&- 1 - 2F_{n+4} + 2 + 2 - \binom{n+1}{2} = nF_{n+4} - F_{n+5} + 5 - \binom{n+1}{2}
\end{aligned}$$

$\boxed{\text{F16}}$ Spočítajte sumu

$$\begin{aligned}
S_n &= \sum_{0\le i,k\le n} F_{i+k} = \sum_{k=0}^{n}\sum_{i=0}^{n} F_{i+k} = \sum_{k=0}^{n}\sum_{i=k}^{n+k} F_i = \sum_{k=0}^{n}\left[\sum_{i=0}^{n+k} F_i - \sum_{i=0}^{k-1} F_i\right] = \\
&= \sum_{k=0}^{n} [F_{n+k+2} - 1 - F_{k+1} + 1] = \sum_{k=0}^{n} F_{n+k+2} - \sum_{k=0}^{n} F_{k+1} = \\
&= \sum_{k=n+2}^{2n+2} F_k - (F_{n+3} - 1) = F_{2n+4} - 2F_{n+3} + 1.
\end{aligned}$$

$\boxed{\text{F17}}$ Spočítajte sumu

$$S_n = \sum_{0\le i,k\le n} F_{i-k} = \sum_{i=0}^{n}\sum_{k=0}^{i} F_k = \sum_{i=0}^{n} (F_{i+2} - 1) = F_{n+4} - n - 3.$$

$\boxed{\text{F18}}$ Spočítajte sumu

$$S_n = \sum_{0\le i\le k\le n} F_{i+k}$$

$\boxed{\text{F19}}$ Spočítajte sumu

$$S_n = \sum_{0\le i<k\le n} F_{i+k}$$

$\boxed{\text{F20}}$ Spočítajte sumu

$$S_n = \sum_{0\le i<k\le n} F_{k-i}$$

$\boxed{\text{F21}}$ Spočítajte sumu

$$\begin{aligned} S_n &= \sum_{0\le k\le n} 2^k \cdot F_k = \sum_{k=0}^{n} 2^{n-k} F_{n-k} = 2^n \cdot \sum_{k=0}^{n} 2^{-k} F_{n-k} = 2^n \cdot [z^n] \frac{z}{(1-z-z^2)(1-z/2)} = \\ &= 2^n \cdot [z^n] \left[\frac{-2/5}{1-z/2} + \frac{4z+2}{1-z-z^2} \right] = -\frac{2}{5} + 2^{n+1} \frac{F_{n+2}+F_n}{5}. \end{aligned}$$

$\boxed{\text{F22}}$ Spočítajte sumu

$$\begin{aligned} S_n &= \sum_{0\le k\le n} 3^k F_k = \sum_{k=0}^{n} 3^{n-k} F_{n-k} = 3^n \cdot \sum_{k=0}^{n} \left(\frac{1}{3}\right)^k F_{n-k} = \\ &= 3^n [z^n] \left(\frac{z}{(1-z/3)(1-z-z^2)} \right) == -\frac{3}{11} + 3^n \frac{9F_n + 3F_{n+1}}{11}. \end{aligned}$$

$\boxed{\text{F22}}$ Spočítajte sumu

$$\begin{aligned} S_n &= \sum_{0\le k\le n} 3^k \cdot F_{k+1} = \frac{1}{3} \cdot \sum_{0\le k\le n} 3^{k+1} F_{k+1} = \frac{1}{3} \sum_{1\le k\le n+1} 3^k F_k = 3^n F_{n+1} + \frac{1}{3} \sum_{0\le k\le n} 3^k F_k. \\ S_n &= 3^n \frac{F_{n+6} + 2F_{n+4}}{11)} - \frac{1}{11}. \end{aligned}$$

$\boxed{\text{F23}}$ Spočítajte sumu ($m \in \mathbb{N}$, a je konštanta)

$$S_n = \sum_{0\le k\le n} a^k \cdot F_{k+m}$$

6 Kombinatorické sumy

Ak nie je povedané ináč, $n \in \mathbb{N}$.

$\boxed{\text{SC1}}$

$$S_n = \sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} \cdot 1^k \cdot 1^{n-k} = (1+1)^n = 2^n.$$

$\boxed{\text{SC2}}$

$$S_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} \cdot (-1)^k \cdot 1^{n-k} = (1-1)^n = (n=0).$$

SC3

$$S_n = \sum_{k=0}^{n} k\binom{n}{k} = n \cdot \sum_{k=1}^{n}\binom{n-1}{k-1} = n \cdot \sum_{k=0}^{n-1}\binom{n-1}{k} = n \cdot 2^{n-1}.$$

SC4 Vypočítajte nasledujúcu sumu S_n, kde $n \geq 2$

$$\begin{aligned} S_n &= \sum_{k=0}^{n} k(k-1)\binom{n}{k} = \sum_{k=2}^{n} k(k-1)\binom{n}{k} = n(n-1) \cdot \sum_{k=2}^{n}\binom{n-2}{k-2} = \\ &= n(n-1) \cdot \sum_{k=0}^{n-2}\binom{n-2}{k} = n(n-1) \cdot 2^{n-2}. \end{aligned}$$

SC5 Vypočítajte nasledujúcu sumu $S_{m,n}$, kde n, m $n \geq m$ sú prirodzené čísla

$$\begin{aligned} S_{m,n} &= \sum_{m \leq k \leq n} (-1)^k \binom{n}{k}\binom{k}{m} = \sum_{m \leq k \leq n} (-1)^k \binom{n}{m}\binom{n-m}{k-m} = \\ &= \binom{n}{m} \cdot \sum_{m \leq k+m \leq n} (-1)^{k+m}\binom{n-m}{k} = \binom{n}{m}(-1)^m \cdot \sum_{0 \leq k \leq n-m} (-1)^k \binom{n-m}{k} = \\ &= \binom{n}{m}(-1)^m(n-m=0) = (-1)^m \cdot \delta_{n,m}. \end{aligned}$$

SC6

$$S = \sum_{k \geq 1} \frac{k-1}{k!} = \sum_{k \geq 1} \frac{1}{(k-1)!} - \frac{1}{k!} = \sum_{k \geq 0} \frac{1}{k!} - \sum_{k \geq 1} \frac{1}{k!} = 1.$$

SC7

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}\binom{n}{k} = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}\left[\binom{n-1}{k} + \binom{n-1}{k-1}\right] = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}\binom{n-1}{k} + \\ &+ \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}\binom{n-1}{k-1} = \sum_{k=1}^{n-1} \frac{(-1)^{k-1}}{k}\binom{n-1}{k} + \frac{1}{n}\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k} = \\ &= \sum_{k=1}^{n-1} \frac{(-1)^{k-1}}{k}\binom{n-1}{k} + \frac{1}{n}\left[-\sum_{k=0}^{n}(-1)^k\binom{n}{k} + (-1)^0\binom{n}{0}\right] = S_{n-1} + \frac{1}{n}; \\ S_n &= H_n; \qquad n \neq 0. \end{aligned}$$

SC8

$$\sum_{k=1}^{n} k \cdot k! = \sum_{k=1}^{n}(k+1) \cdot k! - \sum_{k=1}^{n} k! = \sum_{k=1}^{n}(k+1)! - \sum_{k=1}^{n} k! = \sum_{k=2}^{n+1} k! - \sum_{k=1}^{n} k! = (n+1)! - 1.$$

$\boxed{\text{SC9}}$

$$\begin{aligned}
S_n &= \sum_k \binom{n}{k} \frac{k!}{(n+1+k)!} = \sum_{k\geq 0} \frac{n!k!}{k!(n-k)!(n+1+k)!} = \frac{n!}{(2n+1)!} \sum_{k\geq 0} \frac{(2n+1)!}{(n-k)!(n+1+k)!} = \\
&= \frac{n!}{(2n+1)!} \sum_{k\geq 0} \frac{(2n+1)!}{(n-k)!(n+1+k)!} = \frac{n!}{(2n+1)!} \sum_{0\leq k\leq n} \binom{2n+1}{n-k} = \\
&= \frac{n!}{(2n+1)!} \sum_{0\leq k\leq n} \binom{2n+1}{k} = \frac{n!}{(2n+1)!} \cdot 2^{2n}.
\end{aligned}$$

$\boxed{\text{SC10}}$

$$\begin{aligned}
S_n &= \sum_k (-1)^k \binom{r-k}{n-k} \binom{r}{k} = \sum_k (-1)^{k+n-k} \binom{n-k-r+k-1}{n-k} \binom{r}{k} = \\
&= (-1)^n \sum_k \binom{n-r-1}{n-k} \binom{r}{k} = (-1)^n \binom{n-1}{n} = (n=0)
\end{aligned}$$

$\boxed{\text{SC11}}$

$$\begin{aligned}
S_n &= \sum_k \binom{n}{3k} = \frac{(1+1)^n + (1+\omega_1)^n + (1+\omega_2)^n}{3} = \\
&= \frac{1}{3}\left[2^n + 2\cos\frac{2\pi n}{3} \cdot (-1)^n\right] \quad \omega_1 = e^{2\pi i/3}, \omega_2 = e^{4\pi i/3} \\
(1+\omega_1)^n &= \left(1 + \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^n = \left(1 - 1/2 + \frac{i\sqrt{2}}{2}\right)^n = e^{\pi i n/3} \\
(1+\omega_2)^n &= \left(1 + \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)^n = \left(1 - 1/2 - \frac{i\sqrt{2}}{2}\right)^n = e^{5\pi i n/3} \\
(1+\omega_1)^n \cdot \omega_1 &= \exp\frac{\pi i(n+2)}{3} \\
(1+\omega_1)^n \cdot \omega_2 &= \exp\frac{\pi i(n+4)}{3} \\
(1+\omega_2)^n \cdot \omega_1 &= \exp\frac{\pi i(5n+2)}{3} \\
(1+\omega_2)^n \cdot \omega_2 &= \exp\frac{\pi i(5n+4)}{3}
\end{aligned}$$

$$
\begin{aligned}
(1-\omega_1)^n &= \left(1-\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}\right)^n = \left(1+1/2-\frac{i\sqrt{2}}{2}\right)^n = 3^{n/2}e^{11\pi i n/6} \\
(1-\omega_2)^n &= \left(1-\cos\frac{4\pi}{3} - i\sin\frac{4\pi}{3}\right)^n = \left(1+1/2+\frac{i\sqrt{2}}{2}\right)^n = 3^{n/2}e^{\pi i n/6} \\
(1-\omega_1)^n\cdot\omega_1 &= 3^{n/2}\exp\frac{\pi i(11n+4)}{6} \\
(1-\omega_1)^n\cdot\omega_2 &= 3^{n/2}\exp\frac{\pi i(11n+8)}{6} \\
(1-\omega_2)^n\cdot\omega_1 &= 3^{n/2}\exp\frac{\pi i(n+4)}{6} \\
(1+\omega_2)^n\cdot\omega_2 &= 3^{n/2}\exp\frac{\pi i(n+8)}{6} \\
(1+\omega_1)^n &+ (1+\omega_2)^n = \cos\frac{n\pi}{3} + \cos\frac{5n\pi}{3} + i\sin\frac{5n\pi}{3} + i\sin\frac{n\pi}{3} = \\
&= 2\cos\frac{6n\pi}{6}\cdot\cos\frac{4\pi n}{6} + 2i\sin\frac{6n\pi}{6}\cdot\cos\frac{4\pi n}{6} = 2\cos\frac{2\pi n}{3}\cdot(-1)^n
\end{aligned}
$$

$\boxed{\text{SC12}}$

$$
\begin{aligned}
S_n &= \sum_k \binom{n}{3k+1} = \frac{(1+1)^n + \omega_1^2(1+\omega_1)^n + \omega_2^2(1+\omega_2)^n}{3} = \\
&= \frac{1}{3}\left[2^n + (-1)^{n+1}\cdot 2\cdot\cos\frac{(2n-1)\pi}{3}\right] \quad \omega_1 = e^{2\pi i/3}, \omega_2 = e^{4\pi i/3}
\end{aligned}
$$

$\boxed{\text{SC13}}$

$$
\begin{aligned}
S_n &= \sum_k \binom{n}{3k+2} = \frac{(1+1)^n + \omega_1(1+\omega_1)^n + \omega_2(1+\omega_2)^n}{3} = \\
&= \frac{1}{3}\left[2^n + (-1)^{n}\cdot 2\cdot\cos\frac{(2n+1)\pi}{3}\right] \quad \omega_1 = e^{2\pi i/3}, \omega_2 = e^{4\pi i/3}
\end{aligned}
$$

$\boxed{\text{SC14}}$

$$
\begin{aligned}
S_n &= \sum_k \binom{n}{3k}(-1)^k = \frac{(1-1)^n + (1-\omega_1)^n + (1-\omega_2)^n}{3} = \\
&= 2\cdot 3^{n/2-1}\cos\frac{\pi n}{6}.
\end{aligned}
$$

$\boxed{\text{SC15}}$

$$
\begin{aligned}
S_n &= \sum_k \binom{n}{3k+1}(-1)^k = \frac{(1-1)^n - (1-\omega_1)^n\omega_2 + (1-\omega_2)^n\omega_1}{3} = \\
&= 2\cdot 3^{n/2-1}\cdot(-1)^n\cos\frac{\pi(5n+2)}{6}.
\end{aligned}
$$

SC16

$$\begin{aligned} S_n &= \sum_k \binom{n}{3k+2}(-1)^k = \frac{(1-1)^n - (1-\omega_1)^n\omega_1 + (1-\omega_2)^n\omega_2}{3} = \\ &= 2\cdot 3^{n/2-1}\cdot(-1)^{n+1}\cos\frac{\pi(5n-2)}{6}. \end{aligned}$$

SC17

$$\begin{aligned} S_n &= \sum_k \binom{n}{5k+1} = \frac{(1+1)^n + \omega_4(1+\omega_1)^n + \omega_3(1+\omega_2)^n + \omega_2(1+\omega_3)^n + \omega_1(1+\omega_4)^n}{5} = \\ &= \frac{1}{5}\,[2^n + \dots)] \quad \omega_1 = e^{2\pi i/5}, \omega_2 = e^{4\pi i/5}, \omega_3 = e^{6\pi i/5}, \omega_4 = e^{8\pi i/5} \end{aligned}$$

SC18

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k+1}\binom{n}{k} = \frac{1}{n+1}\sum_{k=1}^{n}(-1)^{k+1}\binom{n+1}{k+1} = \frac{1}{n+1}\sum_{k=2}^{n+1}(-1)^k\binom{n+1}{k} = \\ &= \frac{1}{n+1}\left[\sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k} - n\right] = \frac{n}{n+1} \end{aligned}$$

SC19

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k(k+1)}\binom{n}{k} = \sum_{k=1}^{n}\left[\frac{(-1)^{k+1}}{k} - \frac{(-1)^{k+1}}{(k+1)}\right]\binom{n}{k} = \sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}\binom{n}{k} \\ &- \sum_{k=1}^{n}\frac{(-1)^{k+1}}{k+1}\binom{n}{k} = H_n - \frac{n}{n+1} \end{aligned}$$

SC20

$$S_n = \sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k} = \sum_{k=0}^{n}\frac{n!}{(k+1)!(n-k)!} = \frac{1}{n+1}\left[\sum_{k=0}^{n}\binom{n+1}{k} - \binom{n+1}{n+1}\right] = \frac{2^{n+1}-1}{n+1}.$$

SC21 Upravte

$$\begin{aligned} \binom{-1/2}{n} &= \frac{(-1/2)^{\underline{n}}}{n!} = (-1)^n\frac{\frac{1}{2}\cdot\frac{3}{2}\dots\frac{2n-1}{2}}{n!} = (-1)^n\frac{\frac{1}{2}\cdot\frac{2}{2}\cdot\frac{3}{2}\cdot\frac{4}{2}\dots\frac{2n-1}{2}\cdot\frac{2n}{2}}{n!n!} = \\ &= \frac{(-1)^n}{2^{2n}}\frac{(2n)!}{(n!)^2} = \frac{(-1)^n}{2^{2n}}\cdot\binom{2n}{n}. \end{aligned}$$

SC22 Upravte

$$\binom{1/2}{n} = \frac{(-1/2}{n}\cdot\binom{-1/2}{n-1} = \frac{(-1)^{n-1}}{2^{2n-1}\cdot n}\cdot\binom{2n-2}{n-1}.$$

SC23 Upravte

$$\binom{1/2}{n+1} = \frac{(-1)^n}{2^{2n+1}\cdot(n+1)}\cdot\binom{2n}{n}.$$

7 Generujúce funkcie

Táta kapitola obsahuje príklady na vytvárajúce funkcie. Je rozdelená na časti venované odvodzovaniu vytvárajúcich funkcií, výpočtu kombinatorických súm a dokazovaniu kombinatorických identít pomocou vytvárajúcich funkcií. (Tematicky do tejto oblasti patrí aj riešenie rekurentných vzťahov, ale vzhľadom na dôležitosť tejto problematiky sme jej venovali samostatnú kapitolu.) Obsahuje niektoré úlohy, ktoré boli riešené v predchádzajúcich častiach pomocou iných metód; pretože jej cieľom nie je riešenie konkrétnych úloh, ale demonštrácia metód.

7.1 Konštrukcia generujúcich funkcií pre postupnosti čísel

Zápis $(zD)f(z)$ označuje funkciu, ktorú dostaneme vynásobením deriváce funkcie $f(z)$ premennou z; zápis $(zD^k)f(z)$ označuje k násobné použitie operátora zD (zderivuj, výsledok vynásob z, zderivuj, výsledok vynásob z,...).

GF1 Nájdite vytvárajúce funkcie pre nasledujúce postupnosti

$$\{1\}_k \leftrightarrow \sum_k z^k = \frac{1}{1-z}$$

GF2

$$\{k\}_k \leftrightarrow \sum_k kz^k = (zD)\frac{1}{1-z} = \frac{z}{1-z)^2}$$

GF3 Fibonacciho čísla $\{F_k\}_k$

$$\begin{aligned}
F_{n+2} &= F_{n+1} + F_n,\ n \geq 0; \quad F_0 = 0, F_1 = 1. \\
\frac{F(z)-z}{z^2} &= \frac{F(z)}{z} + F(z); \\
F(z) &= \frac{z}{1-z-z^2} = \frac{1}{\sqrt{5}}\left(\frac{1}{1-\Phi_1 z} - \frac{1}{1-\Phi_2 z}\right); \\
&\Phi_1 = \frac{1+\sqrt{5}}{2},\ \Phi_2 = \frac{1-\sqrt{5}}{2}
\end{aligned}$$

GF4 Fibonacciho čísla s párnym indexom $\{F_{2k}\}_k$

$$\begin{aligned}
G(z) &= \sum_k F_{2k} z^k; \\
F_{2n} &= [z^{2n}]F(z) = \frac{1}{\sqrt{5}}\left(\Phi_1^{2n} - \Phi_2^{2n}\right) \Rightarrow \\
G(z) &= \frac{1}{\sqrt{5}}\left(\frac{1}{1-\Phi_1^2 z} - \frac{1}{1-\Phi_2^2 z}\right) = \frac{z}{1-3z+z^2}.
\end{aligned}$$

$\boxed{\text{GF5}}$ Fibonacciho čísla $\{F_{3k}\}_k$

$$\begin{aligned}
G(z) &= \sum_k F_{3k} z^k; \\
F_{3n} &= [z^{3n}]F(z) = \frac{1}{\sqrt{5}}\left(\Phi_1^{3n} - \Phi_2^{3n}\right) \Rightarrow \\
G(z) &= \frac{1}{\sqrt{5}}\left(\frac{1}{1-\Phi_1^3 z} - \frac{1}{1-\Phi_2^3 z}\right) = \frac{2z}{1-4z-z^2}.
\end{aligned}$$

$\boxed{\text{GF6}}$ Konvolúcia Fibonacciho čísel

$$\begin{aligned}
S_n &= \sum_k F_k F_{n-k} = [z^n]F(z)^2 = [z^n]\frac{z^2}{(1-z-z^2)^2} = \left[\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)\right]^2 = \\
&= \left[\frac{1}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{2n} + \left(\frac{1-\sqrt{5}}{2}\right)^{2n}\right)\right] - \left[\frac{2}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)\right]^2
\end{aligned}$$

$\boxed{\text{GF7}}$ Nájdite obyčajnú generujúcu funkciu $F_m(z)$ pre postupnosť $\{F_{mn}\}_{n\geq 0}$, $m \in N$.

$$\begin{aligned}
F_m(z) &= \frac{1}{\sqrt{5}} \times \left(\frac{1}{1-\Phi_1^m \cdot z} - \frac{1}{1-\Phi_2^m \cdot z}\right) = \frac{1}{\sqrt{5}} \times \frac{(\Phi_1^m - \Phi_2^m)z}{1 - z(\Phi_1^m + \Phi_2^m) + (\Phi_1^m \cdot \Phi_2^m)\cdot z^2}. \\
& \quad (\Phi_1^m \cdot \Phi_2^m) = (\Phi_1 \cdot \Phi_2)^m = (-1)^m; \Phi_1 + \Phi_2 = 1. \\
(\Phi_1^m + \Phi_2^m) &= [z^m]\left(\frac{1}{1-\Phi_1 z} + \frac{1}{1-\Phi_2 z}\right) = [z^m]\left(\frac{1-\Phi_2 z + 1 - \Phi_1 z}{(1-\Phi_2 z)(1-\Phi_2 z)}\right) = \\
&= [z^m]\left(\frac{2-(\Phi_2+\Phi_1)z}{1-z-z^2}\right) = [z^m]\left(\frac{2-z}{1-z-z^2}\right) = 2F_{m+1} - F_m = F_m + 2F_{m-1}. \\
(\Phi_1^m - \Phi_2^m) &= [z^m]\left(\frac{1}{1-\Phi_1 z} - \frac{1}{1-\Phi_2 z}\right) = [z^m]\left(\frac{1-\Phi_2 z - 1 + \Phi_1 z}{(1-\Phi_2 z)(1-\Phi_2 z)}\right) = \\
&= [z^m]\left(\frac{(\Phi_1-\Phi_2)z}{1-z-z^2}\right) = [z^m]\left(\frac{z\sqrt{5}}{1-z-z^2}\right) = \sqrt{5}\cdot F_m.
\end{aligned}$$

Potom

$$F_m(z) = \frac{F_m \cdot z}{1-(F_m+2F_{m-1})z + (-1)^m z^2}.$$

$\boxed{\text{GF8}}$ Nájdite obyčajnú generujúcu funkciu $F_{m,a}(z)$ pre postupnosť $\{F_{mn+a}\}_{n\geq 0}$, $m, a \in N; 0 \leq a < n$.

$$\begin{aligned}
F_{m,a}(z) &= \frac{1}{\sqrt{5}}\left(\Phi_1^{mn+a}z^n - \Phi_2^{mn+a}z^n\right) = \frac{1}{\sqrt{5}}\left(\frac{\Phi_1^a}{1-\Phi_1^m\cdot z} - \frac{\Phi_2^a}{1-\Phi_2^m \cdot z}\right) = \\
&= \frac{1}{\sqrt{5}}\left(\frac{\Phi_1^a(1-\Phi_2^m z) - \Phi_2^a(1-\Phi_1^m z)}{(1-\Phi_1^m z)(1-\Phi_2^m z)}\right) = \frac{1}{\sqrt{5}}\left(\frac{\Phi_1^a - \Phi_2^a + \Phi_1^m\Phi_2^a\cdot z - \Phi_2^m\Phi_1^a \cdot z}{1-z(\Phi_1^m+\Phi_2^m) + (\Phi_1^m\cdot\Phi_2^m)\cdot z^2}\right) = \\
&= \frac{1}{\sqrt{5}}\left(\frac{\Phi_1^a - \Phi_2^a + (\Phi_1\Phi_2)^a \cdot (\Phi_1^{m-a} - \Phi_2^{m-a})\cdot z}{1-(F_m+2F_{m-1})z + (-1)^m z^2}\right)
\end{aligned}$$

$$\frac{1}{\sqrt{5}}(\Phi_1^{m-a} - \Phi_2^{m-a}) = F_{m-a};\ (\Phi_1\Phi_2)^a = (-1)^a;\ \Phi_1^a - \Phi_2^a = \sqrt{5}\cdot F_a.$$

$$F_{m,a}(z) = \frac{F_a + (-1)^a F_{m-a}\cdot z}{1 - (F_m + 2F_{m-1})z + (-1)^m z^2}$$

7.2 Výpočty súm pomocou generujúcich funkcií

Budeme využívať dve základné a jednu pokročilejšiu techniku.

1. Nech je daná suma $S_n = \sum_{k=0}^n a_k$. Ak poznáme generujúcu funkciu $A(x)$ pre postupnosť $\{a_n\}_n$, tak $S_n = [x^n]A(x)/(1-x)$.
2. Nech je daná suma $S_n = \sum_{k=0}^n a_k b_{n-k}$.Ak poznáme generujúce funkcie $A(x)$ pre postupnosť $\{a_n\}_n$ a $B(x)$ pre postupnosť $\{b_n\}_n$,tak $S_n = [x^n]A(x)B(x)$. (Všimnite si, že prvá metóda je len špeciálnym prípadom druhej, $b_n = 1, n \in N$ a $B(x) = 1/(1 = x)$.
3. Snake oil method (Wilf, [2])
   - Nech je daná suma $S_n = \sum_{k=0}^n a_k$. Predpokladáme, že existuje obyčajná vytvárajúca funkcia $S(x) \leftrightarrow \{S_n\}_n$; $S(x) = \sum_n S_n x^n$.
   - Vynásobíme obe strany rovnosti výrazom x^n a sčítame cez všetky n. Na ľavej strane rovnosti máme vytvárajúcu funkciu $S(x)$ a na pravej dvojitú sumu.
   - Zmeníme poradie sumácie a sumujeme najprv cez n a potom cez k.
   - Určíme (ak sa podarí) koeficient pri člene x^n a určíme S_n.

$\boxed{\text{GF8}}$ $S_n = \sum_{k=0}^n 1$.

$$S_n = [x^n]\frac{x}{(1-x)}\cdot\frac{1}{1-x} = [x^n]\frac{x}{(1-x)^2} = \binom{n+1}{1} = n+1.$$

$\boxed{\text{GF9}}$ $S_n = \sum_{k=0}^n (-1)^k$.

$$S_n = [x^n]\frac{x}{(1-x)}\cdot\frac{1}{1+x} = \frac{1}{2}[x^n]\frac{x}{(1-x)} + \frac{1}{1+x} = \frac{1}{2}(1+(-1)^n).$$

$\boxed{\text{GF10}}$ $S_n = \sum_{k=0}^n k$.

$$S_n = [x^n]\frac{x}{(1-x)^2}\cdot\frac{1}{1-x} = [x^n]\frac{x}{(1-x)^3} = \binom{n+1}{2}.$$

$\boxed{\text{GF11}}$ $S_n = \sum_{k=0}^n (-1)^k k$.

$$S_n = [x^n]\frac{-x}{(1+x)^2}\cdot\frac{1}{1-x} = [x^n]\frac{1}{2(1+x)^2} - \frac{1}{4(1-x)} - \frac{1}{4(1+x)} = \frac{n+1}{2} - \frac{1}{4} - \frac{(-1)^n}{4}.$$

$\boxed{\textbf{GF12}}$ $S_n = \sum_{k=0}^{n} k2^k$.

$$\begin{aligned} S_n &= \sum_{k=0}^{n}(n-k)2^{n-k} = 2^n \cdot \sum_{k=0}^{n}(n-k)2^{-k} \\ S_n &= 2^n \cdot [x^n]\frac{x}{(1-x)^2} \cdot \frac{1}{1-\frac{1}{2}x} = 2^n \cdot [x^n]\left(\frac{2}{(1-x)^2} - \frac{4}{1-x} + \frac{2}{1-\frac{1}{2}x}\right) = \\ &= 2^{n+1}(n+1) - 2^{n+2} + 2 = 2^{n+1}(n-1) + 2. \end{aligned}$$

$\boxed{\textbf{GF13}}$ $S_n = \sum_{k=0}^{n}(-1)^k k2^k$.

$$\begin{aligned} S_n &= \sum_{k=0}^{n}(n-k)(-2)^{n-k} = (-2)^n \cdot \sum_{k=0}^{n}(n-k)(-2)2^{-k} \\ S_n &= (-2)^n \cdot [x^n]\frac{x}{(1-x)^2} \cdot \frac{1}{1+\frac{1}{2}x} = (-2)^n \cdot [x^n]\left(\frac{2}{3(1-x)^2} - \frac{4}{9(1-x)} - \frac{2}{9(1+\frac{1}{2}x)}\right) = \\ &= \frac{(-1)^n}{9}\left((3n+1)\cdot 2^{n+1} - 2\cdot(-1)^n\right). \end{aligned}$$

$\boxed{\textbf{GF14}}$ $S_n = \sum_{k=0}^{n} F_k$.

$$S_n = [x^n]\frac{x}{1-x-x^2} \cdot \frac{1}{1-x} = [x^n]\frac{x+1}{1-x-x^2} - \frac{1}{1-x} = F_{n+2} - 1.$$

$\boxed{\textbf{GF15}}$ $S_n = \sum_{k=0}^{n}(-1)^k F_k$.

$$\begin{aligned} S_n &= \sum_{k=0}^{n}(-1)^{n-k}F_{n-k} = (-1)^n \cdot \sum_{k=0}^{n}(-1)^{-k}F_{n-k} \\ S_n &= (-1)^n \cdot \frac{x}{1-x-x^2} \cdot \frac{1}{1+x} = (-1)^n \cdot [x^n]\frac{1-x}{1-x-x^2} - \frac{1}{1-x} = (-1)^n F_{n-1} - 1. \end{aligned}$$

$\boxed{\textbf{GF16}}$

$$\begin{aligned} S_n &= \sum_{k=0}^{n}(n-k)\cdot F_k = [x^n]\left(\frac{x}{1-x-x^2} \cdot \frac{x}{(1-x)^2}\right) = [x^n]\left(\frac{2+x}{1-x-x^2} - \frac{1}{1-x} - \frac{1}{(1-x)^2}\right) = \\ &= 2F_{n+1} + F_n - n - 2 = F_{n+3} - n - 2. \end{aligned}$$

$\boxed{\textbf{GF17}}$ $S_n = \sum_{k=0}^{n} kF_k$.

1. riešenie (zmena poradia sumácie; $k \leftarrow n-k$):

$$\begin{aligned} S_n &= \sum_{k=0}^{n}(n-k)F_{n-k} = n \cdot \sum_{k=0}^{n} F_k - \sum_{k=0}^{n} kF_{n-k} = n \cdot \sum_{k=0}^{n} F_k - \sum_{k=0}^{n}(n-k)F_k = \\ &= nF_{n+2} - n - F_{n+3} + n + 2 = nF_{n+2} - F_{n+3} + 2. \end{aligned}$$

2. riešenie

$$\begin{aligned}
F(x) &= \sum_{k\geq 0} F_k x^k = \frac{x}{1-x-x^2}\\
xDF(x) &= \frac{x+x^3}{(1-x-x^2)^2} = \sum_{k\geq 0} kF_k \cdot x^k \Leftrightarrow \{k\cdot F_k\}_{k\geq 0};\\
xDF(x)\cdot\frac{1}{1-x} &= \frac{x+x^3}{(1-x-x^2)^2(1-x)} \Leftrightarrow \{\sum_{k=0}^{n} k\cdot F_k\}_{n\geq 0}\\
S_n &= [x^n]\frac{x+x^3}{(1-x-x^2)^2(1-x)} = [x^n]\left[\frac{2x+1}{(1-x-x^2)^2} - \frac{2x+3}{1-x-x^2} + \frac{2}{1-x}\right].
\end{aligned}$$

$\boxed{\text{GF18}}$

$$\begin{aligned}
S &= \sum_{n\geq 0}\frac{3n^2-2n+5}{n!} = \sum_{n\geq 0}\frac{3n^2}{n!} - \sum_{n\geq 0}\frac{2n}{n!} + \sum_{n\geq 0}\frac{5}{n!}\\
\hat{S}(z) &= \sum_{n\geq 0}\frac{3n^2\cdot z^n}{n!} - \sum_{n\geq 0}\frac{2n\cdot z^n}{n!} + \sum_{n\geq 0}\frac{5\cdot z^n}{n!} = \{3(zD)^2 - 2(zD) + 5\}e^z =\\
&= 3(z^2+z)e^z - 2ze + 5e^z.\\
S &= \hat{S}(1) = 9e.
\end{aligned}$$

$\boxed{\text{GF19}}$

$$\begin{aligned}
S_n &= \sum_{k=0}^{n} k\cdot F_{2k} = -\sum_{k=0}^{n}(n-k)\cdot F_{2k} + n\cdot\sum_{k=0}^{n} F_{2k} = [z^n]G(z)\\
G(z) &= \left(-\frac{z}{(1-z)^2}\cdot\frac{z}{(1-3z+z^2)} + n\cdot\frac{z}{(1-3z+z^2)}\cdot\frac{1}{1-z}\right) =\\
&= \frac{-z^2-nz^2+nz}{(1-3z+z^2)\cdot(1-z)^2} = \frac{1}{(1-z)^2} - \frac{n+1}{1-z} - \frac{z(n+1}{1-3z+z^2} + \frac{n}{1-3z+z^2}\\
S_n &= (n+1)-(n+1)-(n+1)F_{2n} + nF_{2n+2} = nF_{2n+1} - F_{2n}.
\end{aligned}$$

$\boxed{\text{GF20}}$

$$\begin{aligned}
S_n &= \sum_{k=0}^{n} F_{3k}\\
S_n &= [z^n]\frac{2z}{(1-4z-z^2)(1-z)} = [z^n]\frac{1}{2}\left[\frac{z+1}{1-4z-z^2} - \frac{1}{1-z}\right] = \frac{F_{3n+2}-1}{2}.
\end{aligned}$$

$\boxed{\text{GF21}}$ **Nájdite o.g.f. postupnosti** $\{3n^2-2n+7\}_{n\geq 0}$.

$$F(z) = \left[3\left((zD)^2\right) - 2(zD) + 7\right]\frac{1}{1-z} = 3\cdot\frac{z^2+z}{(1-z)^3} - 2\cdot\frac{z}{(1-z)^2} + \frac{7}{1-z}.$$

$\boxed{\text{GF22}}$

$$\begin{aligned} S_n &= \sum_{k=0}^{n} k^2 = [z^n]\frac{1}{1-z}(zD)^2\frac{1}{1-z} = [z^n]\frac{z+z^2}{(1-z)^4} = \binom{n+2}{3} + \binom{n+1}{3} = \\ &= \frac{n(n+1)(2n+1)}{6}. \end{aligned}$$

$\boxed{\text{GF23}}$

$$S_n = \sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}. \qquad (1+z)^n = \sum_{k=0}^{n} \binom{n}{k} z^k.$$

$$\begin{aligned} \int_0^1 (1+z)^n dz &= \frac{(1+z)^{n+1}}{n+1}\Big|_0^1 = \frac{2^{n+1}-1}{n+1}; \\ \int_0^1 \sum_{k=0}^{n} \binom{n}{k} z^k dz &= \sum_{k=0}^{n} \binom{n}{k} \int_0^1 z^k dz = \sum_{k=0}^{n} \binom{n}{k} \frac{z^{k+1}}{k+1}\Big|_0^1 = \sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}. \end{aligned}$$

8 Rekurentné vzťahy

$\boxed{\text{R1}}$

$$a_0 = 7, \quad 2a_n = na_{n-1} + 2n! \quad n > 0; \quad s_n = \frac{2^{n-1}}{n!}, \quad T_n = \frac{2^n}{n!}a_n$$

$$\begin{aligned} T_n &= T_{n-1} + 2^n, \quad n > 0, \quad T_0 = a_0 = 7 \\ T_n &= T_0 + \sum_{k=1}^{n} 2^k = 7 + 2^{n+1} - 2 = 2^{n+1} + 5 \\ a_n &= 2n! + \frac{5n!}{2^n} \end{aligned}$$

$\boxed{\text{R2}}$

$$a_n = 2a_{n-1} - a_{n-2} + (-1)^n; \quad n > 1; \quad a_0 = a_1 = 1$$

$$a_{n+2} - 2a_{n+1} + a_n = (-1)^n \qquad A(z) = \sum_k a_k z^k$$

$$\frac{A(z) - 1 - z}{z^2} - 2\frac{A(z) - 1}{z} + A(z) = \frac{1}{1+z} \qquad |.z^2$$

$$A(z) = \frac{1}{(1-z)^2(1+z)} = \frac{1/2}{(1-z)^2} + \frac{1/4}{1-z} + \frac{1/4}{1+z}$$

$$a_n = \frac{n+1}{2} + \frac{1+(-1)^n}{4}$$

$\boxed{\text{R3}}$

$$a_n = 2a_{n-1} - a_{n-2} + (-1)^n; \quad n > 1; \quad a_0 = 0, \ a_1 = 1$$

$$A(z) = \frac{2z^2+z}{(1+z)(1-z)^2} = \frac{3}{2(1-z)^2} - \frac{7}{4(1-z)} + \frac{1}{4(1+z)}$$

$$a_n = \frac{3}{2}(n+1) - \frac{7}{4} + \frac{(-1)^n}{4}$$

$\boxed{\text{R4}}$ **Riešte rekurentný vzťah**

$$a_n = 2a_{n-1} - a_{n-2} + (-1)^n + n \quad n > 1, \quad a_0 = 1, a_1 = 0$$

$$\begin{aligned}
A(z) &= \frac{1}{(1+z)(1-z)^2} + \frac{z}{(1-z)^4} - \frac{2z}{(1-z)^2} = \\
&= \frac{1/4}{1+z} + \frac{1/2}{(1-z)^2} + \frac{1/4}{1-z} + \frac{z}{(1-z)^4} - \frac{2z}{(1-z)^2} \\
a_n &= \frac{(-1)^n}{4} + \frac{1}{2(n+1)} + \frac{1}{4} + \binom{n+2}{3} - 2n = \frac{2n^3+6n^2-14n+3\cdot(-1)^n+9}{12}.
\end{aligned}$$

$\boxed{\text{R5}}$

$$\begin{aligned}
a_0 &= a_1 = 1 \\
a_n &= a_{n-1} - 2a_{n-2} + 1 + (-1)^n + n \quad n > 1
\end{aligned}$$

$$A(z) = \frac{1}{2\,(1+z)} - \frac{11}{27\,(1+2\,z)} - \frac{1}{3\,(1-z)^3} + \frac{4}{9\,(1-z)^2} - \frac{11}{54\,(1-z)}$$

$$a_n = \frac{(-1)^n}{2} - \frac{11}{27}\cdot(-1)^n 2^n - \frac{1}{3}\binom{n+2}{2} + \frac{4}{9}(n+1) - \frac{11}{54}$$

$\boxed{\text{R6}}$

$$\begin{aligned}
a_0 &= a_1 = 1 \\
a_n &= 2a_{n-1} + 3a_{n-2} + n \quad n > 1
\end{aligned}$$

$$A(z) = \frac{-1}{4(1-z)^2} - \frac{1}{4(1-z)} + \frac{11}{16(1+z)} + \frac{13}{16(1-3z)}$$

$$a_n = \frac{-1}{4}(n+1) - \frac{1}{4} + \frac{11}{16}(-1)^n + \frac{13}{16}3^n$$

$\boxed{\text{R7}}$

$$\begin{aligned}
a_0 &= 1, a_1 = 2 \\
a_{n+2} &= 3a_{n+1} - 2a_n + 3 \quad n > 0
\end{aligned}$$

$$A(z) = \frac{4}{1-2z} - \frac{3}{(1-z)^2}$$

$$a_n = 4\cdot 2^n - 3n - 3.$$

$\boxed{\text{R8}}$

$$\begin{aligned}
a_0 &= 1, a_1 = 3 \\
a_{n+2} &= 3a_{n+1} - 2a_n + n \quad n > 0
\end{aligned}$$

$$A(z) = \frac{1-2z+z^2+z^3}{(1-z)^3(1-2z)} = \frac{3}{1-2z} - \frac{1}{(1-z)} + \frac{-z^2+z-1}{(1-z)^3}$$

$$a_n = 3 \cdot 2^n - 1 - \binom{n}{2} + \binom{n+1}{2} - \binom{n+2}{2} = 3 \cdot 2^n - \frac{n^2+n}{2} - 2.$$

$\boxed{\text{R9}}$ Určte zložitosť obvodu počítajúceho prenos pre n-tý rád v sčítačke so zrýchleným prenosom, keď zložitosť je

1. počet literálov vo formule

2. počet operátorov vo formule vyjadrujúcej logickú funkciu daného obvodu.

Formula pre prenos:

$$\begin{aligned} r_0 &= a_0 b_0 \\ r_n &= a_n b_n + a_n r_{n-1} + b_n r_{n-1}, \end{aligned}$$

t.j. zložitosť bude

$$\begin{aligned} s_0 &= 2 \\ s_n &= 2s_{n-1} + 4, \end{aligned}$$

resp.

$$\begin{aligned} t_0 &= 1 \\ t_n &= 2t_{n-1} + 5. \end{aligned}$$

$$\begin{aligned} s_{n+1} &= 2s_n + 4 \\ \frac{S(z)-2}{z} &= 2S(z) + \frac{4}{1-z} \\ S(z) &= \frac{-4}{1-z} + \frac{6}{1-2z} \\ s_n &= [z^n]S(z) = -4 + 6 \cdot 2^n. \end{aligned}$$

Analogicky

$$t_n = -5 + 6 \cdot 2^n.$$

$\boxed{\text{R10}}$ Riešte rekurentný vzťah

$$a_n = 2a_{n-1} - a_{n-2} + 2(-1)^n \quad n > 1, \quad a_0 = 1, a_1 = 3$$

$$\begin{aligned} A(z) &= \frac{1}{2(1+z)} + \frac{3}{(1-z)^2} - \frac{5}{2(1-z)} \\ a_n &= \frac{1}{2}(-1)^n + 3n + \frac{1}{2} \end{aligned}$$

$\boxed{\text{R11}}$ Riešte rekurentný vzťah

$$6a_n = 5a_{n-1} - a_{n-2} + 2^n \quad n > 1, \quad a_0 = 1, a_1 = 1$$

$$\begin{aligned}
A(z) &= \frac{8z^2 - 11z + 6}{(z-2)(z-3)(1-2z)} = \frac{4/3}{1-z/2} + \frac{-3/5}{1-z/3} + \frac{4/15}{1-2z} \\
a_n &= \frac{4}{3} \cdot \left(\frac{1}{2}\right)^n - \frac{3}{5} \cdot \left(\frac{1}{3}\right)^n + \frac{4}{15} \cdot 2^n.
\end{aligned}$$

$\boxed{\text{R12}}$ Riešte rekurentný vzťah

$$a_n = -2na_{n-1} = \sum_k \binom{n}{k} a_k a_{n-k} \quad n > 1, \quad a_0 = 1, a_1 = 1$$

$$\begin{aligned}
a_n &= -2na_{n-1} + \sum_k \binom{n}{k} a_k a_{n-k} + (n = 1) \quad | \times \frac{z^n}{n!} \\
a_n \frac{z^n}{n!} &= -2na_{n-1} \frac{z^n}{n!} + \sum_k \binom{n}{k} a_k a_{n-k} \frac{z^n}{n!} + (n = 1) \frac{z^n}{n!} \quad | \sum_n \\
\sum_n a_n \frac{z^n}{n!} &= -2z \sum_n a_{n-1} \frac{z^{n-1}}{(n-1)!} + \sum_n \sum_k a_k \frac{z^k}{k!} a_{n-k} \frac{z^{n-k}}{(n-k)!} + \sum_n (n = 1) \frac{z^n}{n!} \\
\hat{A}(z) &= -2z\hat{A}(z) + \hat{A}(z)^2 + z \\
0 &= \hat{A}(z)^2 - (2z+1)\hat{A}(z) + z \\
\hat{A}(z) &= \frac{(2z+1) \pm \sqrt{4z^2+1}}{2} \\
a_0 &= \hat{A}(0) = 1 \Longrightarrow \hat{A}(z) = \frac{(2z+1) + \sqrt{4z^2+1}}{2} \\
\hat{A}(z) &= z + \frac{1}{2} + \frac{1}{2} \times (1 + 4z^2)^{1/2} \\
a_n &= \left[\frac{z^n}{n!}\right] \hat{A}(z)
\end{aligned}$$

$$a_n = \begin{cases} 1 & n = 0, 1, \\ 0 & n = 2m+1,\ m > 0; \\ \frac{(2m)! 4^{2m}}{2} \binom{1/2}{m} & n = 2m,\ m \geq 1. \end{cases}$$

$\boxed{\text{R13}}$ Riešte rekurentný vzťah

$$a_{n+1} = \sum_{k=0}^{n} F_k a_{n-k}; \quad a_0 = 1,$$

kde F_k je k-te Fibonacciho číslo. Rekurentný vzťah transformujeme na vzťah medzi vytvárajúcimi funkciami $A(z), F(z)$ a vyjadríme a_n

$$\begin{aligned}
\frac{A(z)-a_0}{z} &= F(z)\cdot A(z) = \frac{z}{1-z-z^2}\cdot A(z);\\
A(z) &= \frac{z^2+z-1}{2z^2+z-1} = \frac{1}{2}+\frac{1/3}{1+z}+\frac{1/6}{1-2z}.\\
a_n &= \frac{1}{2}\cdot(n=0)+\frac{-1^n}{3}+\frac{1}{6}\cdot 2^n.
\end{aligned}$$

$\boxed{\text{R14}}$ Riešte rekurentný vzťah

$$a_{n+1} = \sum_{k=0}^{n}(n-k)a_k; \quad a_0 = 1,$$

$$\begin{aligned}
\frac{A(z)-1}{z} &= \frac{z}{(1-z)^2}\cdot A(z)\\
A(z) &= 1+\frac{z^2}{1-2z};
\end{aligned}$$

$$a_n = \begin{cases} 1 & n=0,\\ 0 & n=1\\ 2^{n-2} & n\geq 2.\end{cases}$$

$\boxed{\text{R15}}$ Riešte rekurentný vzťah

$$a_{n+1} = \sum_{k=0}^{n} k a_k, \quad n>0; \quad a_0 = a_1 = 1.$$

$$\begin{aligned}
a_n &= (n-1)a_{n-1}+(n-2)a_{n-2}+\cdots+a_1, \quad n>1,\\
a_{n-1} &= \phantom{(n-1)a_{n-1}}+(n-2)a_{n-2}+\cdots+a_1, \quad n>2,\\
a_n-a_{n-1} &= (n-1)a_{n-1}\\
a_n &= na_{n-1}; \quad n\geq 3.
\end{aligned}$$

$$a_n = \begin{cases} 1 & n=0,1\\ \frac{n!}{2} & n\geq 2.\end{cases}$$

$\boxed{\text{R16}}$ Riešte rekurentný vzťah

$$a_n = \sum_{k=0}^{n-1} a_k, \quad n>0; \quad a_0 = 1.$$

$$\begin{aligned}
a_n &= a_{n-1}+a_{n-2}+\cdots+a_0, \quad n>0,\\
a_{n-1} &= \phantom{a_{n-1}}+a_{n-2}+\cdots+a_0, \quad n>1,\\
a_n-a_{n-1} &= a_{n-1}\\
a_n &= 2a_{n-1}; \quad n>1
\end{aligned}$$

$$a_n = \begin{cases} 1 & n = 0 \\ 2^{n-1} & n > 0. \end{cases}$$

Riešenie pomocou generujúcich funkcií

$$\begin{aligned} a_{n+1} &= \sum_{k=0}^{n} a_k \\ \frac{A(z)-1}{z} &= \frac{1}{1-z} \cdot A(z) \\ A(z) &= \frac{1+z}{1-2z} = 1 + \frac{z}{1-2z}. \end{aligned}$$

$$a_n = \begin{cases} 1 & n = 0 \\ 2^{n-1} & n > 0. \end{cases}$$

$\boxed{\text{R17}}$ Riešte rekurentný vzťah

$$\begin{aligned} a_n &= 2a_{n-1} - a_{n-2} + n + (-1)^n \quad n > 1, \qquad a_0 = 1, a_1 = -1 \\ A(z) &= \frac{3(1-3z)}{4(1-z)^2} + \frac{1}{4(1+z)} + \frac{2z^2 - z^3}{(1-z)^4} \\ a_n &= \frac{3}{4} - \frac{3}{2} \cdot n + \frac{1}{4} \cdot (-1)^n - \binom{n}{3} + 2\binom{n+1}{3} = \frac{n^3}{6} + \frac{n^2}{2} - \frac{13n}{6} + \frac{3}{4} + \frac{(-1)^n}{4}. \end{aligned}$$

$\boxed{\text{R18}}$ Riešte rekurentný vzťah

$$2a_n = 3a_{n-1} - a_{n-2} + (n-1) \quad n > 1, \quad a_0 = 1, a_1 = -2$$

$$\begin{aligned} 2a_{n+2} &= 3a_{n+1} - a_n + (n+1) \\ 2 \times \frac{A(z) - 1 + 2z}{z^2} &= 3\frac{A(z)-1}{z} - A(z) + \frac{1}{(1-z)^2} \\ A(z) &= \frac{2-7z}{2-3z+z^2} + \frac{z^2}{(2-3z+z^2)(1-z)^2}; \\ a_n &= 1/2 * n^2 - 3/2 * n - 3 + 4 * 2^{-n} \end{aligned}$$

$\boxed{\text{R19}}$ Riešte rekurentný vzťah

$$a_n = 2a_{n-1} - a_{n-2} + F_{n-1} \quad n > 1, \quad a_0 = 1, a_1 = 3$$

$$\begin{aligned} a_{n+2} &= 2a_{n+1} - a_n + F_{n+1} \\ \frac{A(z) - 1 - 3z}{z^2} &= 2 \times \frac{A(z)-1}{z} - A(z) + \frac{1}{1-z-z^2} \\ A(z) &= \frac{1+z}{(1-z)^2} + \frac{z}{(1-z)^2} \times F(z) \\ a_n &= F_{n+3} + n - 1. \end{aligned}$$

$\boxed{\text{R20}}$ Riešte rekurentný vzťah

$$a_n = \sum_{k=1}^{n} (n-k) \cdot a_k + 1; \quad a_0 = 1.$$

$$\begin{aligned}
a_n &= \sum_{k=0}^{n} (n-k) \cdot a_k + 1 - n \cdot a_0 \\
A(z) &= \frac{z}{(1-z)^2} \cdot A(z) - \frac{z}{(1-z)^2} + \frac{1}{1-z} \\
A(z) &= \frac{1-2z}{1-3z+z^2} \\
a_n &= F_{2n+2} - 2F_{2n} = F_{2n-1}.
\end{aligned}$$

$\boxed{\text{R21}}$ Riešte rekurentný vzťah

$$a_n = \sum_{k=0}^{n} (n-k) \cdot a_k + 1; \quad a_0 = 0, a_1 = 1.$$

$$\begin{aligned}
A(z) &= \frac{z}{(1-z)^2} \cdot A(z) + \frac{z}{(1-z)^2} \\
A(z) &= \frac{1-z}{1-3z+z^2} \\
a_n &= F_{2n+2} - F_{2n} = F_{2n}.
\end{aligned}$$

$\boxed{\text{R22}}$ Riešte rekurentný vzťah

$$a_n = \sum_{k} a_{n-k} \cdot a_{k-1}; \quad a_0 = 1, n \neq 0, a_{-k} = 0$$

$$\begin{aligned}
a_n &= \sum_{k=0}^{n} a_{n-k} \cdot a_{k-1} \\
A(z) - 1 &= zA(z)^2
\end{aligned}$$

$$A(z) = \frac{1-\sqrt{1-4z}}{2z}$$

$$a_n = \binom{2n}{n} \times \frac{1}{n+1}.$$

$\boxed{\text{R23}}$ Riešte rekurentný vzťah

$$a_{n+1} = 2 \cdot \sum a_{n-k} \cdot (-1)^k; \qquad a_0 = 1$$

$$\frac{A(z)-1}{z} = 2 \cdot A(z) \frac{1}{1+z}, \quad A(z) = \frac{1+z}{1-z} = 1 + \frac{2z}{1-z}$$

$$a_n = (n = 0) + 2(n > 1).$$

$\boxed{\text{R24}}$ Riešte rekurentný vzťah

$$a_{n+2} = 3a_{n+1} - 2a_n + 1 + n, \qquad a_0 = a_1 = 1.$$

$$\begin{aligned} A(z) &= \frac{1}{1-z} + \frac{z^2}{(1-z)^2(1-2z)} + \frac{z^3}{(1-z)^3(1-2z)} \\ &= \frac{2}{(1-2z)} - \frac{1}{(1-z)^3} \\ a_n &= 2^{n+1} - \binom{n+2}{2}. \end{aligned}$$

$\boxed{\text{R25}}$ Riešte rekurentný vzťah

$$a_{n+2} = 4a_{n+1} - 24a_n + 2n + 1, \qquad a_0 = 1,\ a_1 = 5.$$

$$\begin{aligned} A(z) &= \frac{2}{1-3z} + \frac{1}{1-z} - \frac{z^2}{(1-z)^3} = \frac{2}{(1-2z)} - \frac{1}{(1-z)^3}. \\ a_n &= 2 \cdot 3^n - 1 - \binom{n}{2}. \end{aligned}$$

$\boxed{\text{R26}}$ Riešte rekurentný vzťah

$$\begin{aligned} a_{n+1} &= b_n + a_n,\ n \geq 0; \\ b_{n+1} &= a_{n+1} + b_n,\ n \geq 0; \quad a_0 = 0, b_0 = 1. \\ A(z)/z &= B(z) + A(z); \\ A(z) &= \frac{z}{1-z} B(z); \\ \frac{B(z) - 1}{z} &= \frac{A(z)}{z} + B(z), \\ B(z) &= \frac{1}{1-z} + \frac{A(z)}{1-z}; \\ B(z) &= \frac{1-z}{1-3z+z^2}; \quad A(z) = \frac{z}{1-3z+z^2}; \\ a_n &= F_{2n},\ b_n = F_{2n+1}. \end{aligned}$$

$\boxed{\text{R27}}$ Riešte rekurentný vzťah

$$a_{n+3} = a_{n+2} + a_{n+1} - a_n + n + (-1)^n, \qquad a_0 = 0,\ a_1 = 1,\ a_2 = 1.$$

$$\begin{aligned} A(z) &= \frac{z^5 - z^2 + z}{(1-z)^4(1+z)^2} = \frac{-3}{16(1+z)^2} + \frac{1}{8(1+z)} + \frac{14z^3 - 19z^2 + 8z + 1}{16(1-z)^4} \\ a_n &= \frac{(-1)^n}{8} - \frac{3}{16} \cdot (-1^n) \cdot (n+1) + \frac{1}{16} \cdot \left[14\binom{n}{3} - 19\binom{n+1}{3} + 8\binom{n+2}{3} + \binom{n+3}{3} \right]. \end{aligned}$$

$\boxed{\text{R28}}$ Riešte rekurentný vzťah

$$a_{n+2} = 3a_{n+1} - 2a_n + (-1)^n \cdot n, \qquad a_0 = 1,\ a_1 = -1.$$

$$\begin{aligned} A(z) &= \frac{1 - 2z - 7z^2 - 5z^3}{(1+z)^2(1-z)(1-2z)} = \frac{13}{4(1-z)} - \frac{19}{9(1-2z)} + \frac{1}{6(1+z)^2} - \frac{11}{36(1+z)} \\ a_n &= \frac{13}{4} - \frac{19}{9} \cdot 2^n + \frac{(-1)^n}{6} \cdot (n+1) - \frac{11}{36} \cdot (-1)^n. \end{aligned}$$

$\boxed{\text{R29}}$ Riešte rekurentný vzťah

$$a_{n+2} = 3a_{n+1} - 2a_n + (-1)^n \cdot n, \qquad a_0 = -1,\ a_1 = 1.$$

$$\begin{aligned} A(z) &= \frac{1 + 6z + 9z^2 + 3z^3}{(1+z)^2(1-z)(1-2z)} = \frac{-19}{4(1-z)} - \frac{53}{9(1-2z)} + \frac{1}{6(1+z)^2} - \frac{-11}{36(1+z)} \\ a_n &= \frac{-19}{4} + \frac{53}{9} \cdot 2^n + \frac{(-1)^n}{6} \cdot (n+1) - \frac{11}{36} \cdot (-1)^n. \end{aligned}$$

$\boxed{\text{R30}}$ Riešte rekurentný vzťah

$$a_{n+2} = a_{n+1} + \sum_{k=0}^{n} a_k F_{n-k}, \qquad a_0 = 1,\ a_1 = 2.$$

$$\begin{aligned} A(z) &= \frac{1 - 2z^2 - z^3}{1 - 2z} = \frac{3}{8(1-2z)} - \frac{5}{8} + \frac{5}{4}x + \frac{1}{2}x^2 \\ a_n &= \frac{3}{8}2^n + \frac{5}{8} \cdot (n-0) + \frac{5}{4} \cdot (n=1) + \frac{1}{2} \cdot (n=2). \end{aligned}$$

9 Asymptotické odhady

$\boxed{\text{A1}}$ Odhadnite $\binom{2n}{n}$ s presnosťou $O(n^{-2})$.

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2} = \frac{\sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}\left(1 + \frac{1}{24n} + O(n^{-2})\right)}{2\pi n\left(\frac{n}{e}\right)^{2n}\left(1 + \frac{1}{12n} + O(n^{-2})\right)^2} = \frac{2^{2n}}{\sqrt{\pi n}} \frac{\left(1 + \frac{1}{24n} + O(n^{-2})\right)}{\left(1 + \frac{1}{12n} + O(n^{-2})\right)^2}$$

$$\begin{aligned} &\left(1 + \frac{1}{24n} + O(n^{-2})\right) = \left(1 + \frac{1}{24n}\right)\left(1 + O(n^{-2}\right) \\ &\left(1 + \frac{1}{12n} + O(n^{-2})\right)^{-2} = \left(1 + \frac{1}{12n}\right)^{-2}\left(1 + O(n^{-2})\right)^{-2} \\ &\left(1 + O(n^{-2})\right)^{-2} = \left(1 + O(n^{-2})\right) \\ &\left(1 + \frac{1}{12n}\right)^{-2} = 1 + \binom{-2}{1}\frac{1}{12n} + O(n^{-2}) = \left(1 - \frac{1}{6n}\right)\left(1 + O(n^{-2})\right) \end{aligned}$$

$$\binom{2n}{n} = \frac{2^{2n}}{\sqrt{\pi n}}\left(1 - \frac{1}{8n} + O(n^{-2}\right)$$

$\boxed{\text{A2}}$ Odhadnite $\binom{2n+1}{n}$ s presnosťou $O(n^{-2})$.

$$\begin{aligned}
\binom{2n+1}{n} &= \frac{(2n+1)!}{n!(n+1)!} = \binom{2n}{n}\frac{2n+1}{n+1} \\
\frac{2n+1}{n+1} &= \frac{2\left(1+\frac{1}{2n}\right)}{1+\frac{1}{n}} = 2\left(1+\frac{1}{2n}\right)\left(1-\frac{1}{n}\right)\left(1+O(n^{-2}\right) = 2\left(1-\frac{1}{2n}+O(n^{-2})\right)
\end{aligned}$$

$$\binom{2n+1}{n} = \frac{2^{2n+1}}{\sqrt{\pi n}}\left(1 - \frac{5}{8n} + O(n^{-2}\right)$$

$\boxed{\text{A3}}$ Odhadnite s presnosťou $O(n^{-3})$ podiel

$$\frac{n^{\underline{k}}}{n^k},$$

ak k je konštanta.

$$\begin{aligned}
\frac{n^{\underline{k}}}{n^k} &= \prod_{j=0}^{k=1}\left(1-\frac{j}{n}\right) = 1 - \frac{1}{n}\cdot\sum_{j=0}^{k-1} j + \frac{1}{n^2}\sum_{0\le i<j<k} ij + O\left(\frac{k^6}{n^3}\right); \\
\sum_{j=0}^{k-1} j &= \binom{k}{2}; \qquad \sum_{0\le i,j<k} ij = \sum_{0\le i,<j<k} ij + \sum_{0\le j<i<k} ij + \sum_{0\le j=i<k} ij; \\
\sum_{0\le i,<j<k} ij &= \frac{1}{2}\left[\binom{k}{2}^2 - \frac{k(k-1)(2k-1)}{6}\right] = \frac{3k^4-10k^3+9k^2-2k}{24} \\
\frac{n^{\underline{k}}}{n^k} &= 1 - \binom{k}{2}\cdot\frac{1}{n} + \frac{3k^4-10k^3+9k^2-2k}{24}\cdot\frac{1}{n^2} + O\left(\frac{1}{n^3}\right).
\end{aligned}$$

$\boxed{\text{A3a}}$ Zostrojte asymptotický odhad sumy $\sum_{k\ge 0}\frac{k}{2^k}$

Využijeme (aj v ďalších úlohách) obyčajnú generujúcu funkciu

$$A(x) = \frac{1}{1-ax} = \sum_k a^k x^k,$$

ktorá má polomer konvergencie $1/a$. Aplikujeme na $A(x)$ operátor xD (zderivuj a výsledok vynásob x):

$$xD(A(x)) = \frac{ax}{(1-ax)^2} = \sum_k k a^k x^k,$$

dosadíme $a = 1/2,\ x = 1 < 1/a$:

$$\sum_{k\ge 0} k\frac{1}{2^k} = 2.$$

$\boxed{\text{A3b}}$ Zostrojte asymptotický odhad sumy $\sum_{k\geq 0}\frac{k^2}{2^k}$. Na $A(x)$ aplikujeme operátor $(xD)^2$, do výsledku dosadíme $a = 1/2,\ x = 1 < 1/a$:

$$(xD)^2(A(x)) = \frac{ax + a^2x^2}{(1-ax)^3}; \qquad \sum_{k\geq 0}\frac{k^2}{2^k} = 6.$$

$\boxed{\text{A4}}$ Zostrojte asymptotický odhad sumy

$$\sum_{k=0}^{n}\binom{3n}{k}$$

$$\begin{aligned}
\sum_{k=0}^{n}\binom{3n}{k} &= \sum_{k=0}^{n}\binom{3n}{n-k} = \sum_{0\leq k<3\lg n}\binom{3n}{n-k} + \sum_{k\geq 3\lg n}\binom{3n}{n-k}. \\
\binom{3n}{n-k} &= \binom{3n}{n}\cdot\frac{1}{2^k}\frac{\prod_{j=0}^{k-1}\left(1-\frac{j}{n}\right)}{\prod_{j=1}^{k}\left(1+\frac{j}{2n}\right)} = \binom{3n}{n}\cdot\frac{1}{2^k}\cdot\left[1-\frac{3k^2-k}{4n}+O\left(\frac{k^4}{n^2}\right)\right]. \\
\sum_{k\geq 3\lg n}\binom{3n}{n-k} &= O\left(n\cdot\binom{3n}{n-3\lg n}\right) = \binom{3n}{n}\cdot O\left(n\cdot\frac{1}{2^{3\lg n}}\right) = \binom{3n}{n}\cdot O\left(n^{-2}\right). \\
\sum_{0\leq k<3\lg n}\frac{1}{2^k} &= 2-\frac{1}{2^{3\lg n}} = 2+O(n^{-3}). \\
-\frac{3}{4n}\sum_{0\leq k<3\lg n}\frac{k^2}{2^k} &= \frac{-9}{2n}+O(n^{-3}). \\
+\frac{1}{4n}\sum_{0\leq k<3\lg n}\frac{k}{2^k} &= \frac{1}{2n}+O(n^{-3}). \\
O(n^{-2})\cdot\sum_{0\leq k<3\lg n}\frac{k^4}{2^k} &= O(n^{-2})
\end{aligned}$$

$$\sum_{k=0}^{n}\binom{3n}{k} = \binom{3n}{n}\cdot\left[2-\frac{4}{n}+O(n^{-2})\right]$$

$\boxed{\text{A4a}}$ Zostrojte asymptotický odhad sumy

$$\sum_{k=0}^{2n}\binom{5n}{k}$$

$\boxed{\text{A4b}}$ Zostrojte asymptotický odhad sumy

$$\sum_{k=0}^{\lambda\cdot n}\binom{n}{k} \qquad 0<\lambda<1/2, \quad \lambda\cdot n\in\mathbb{N}.$$

$\boxed{\text{A4c}}$ Zostrojte asymptotický odhad sumy

$$\sum_{k=0}^{n} \binom{5n}{2k}$$

$\boxed{\text{A5}}$

$$\left(1+\frac{3}{n+1}\right)^{2n-1} \qquad O(n^{-2}).$$

$$\left(1+\frac{3}{n+1}\right)^{2n-1} = \left(\frac{n+4}{n+1}\right)^{2n-1} = \left(\frac{1+\frac{4}{n}}{1+\frac{1}{n}}\right)^{2n-1}$$

$$\begin{aligned}
\left(1+\frac{4}{n}\right)^{2n-1} &= \left(1+\frac{4}{n}\right)^{-1}\left(1+\frac{4}{n}\right)^{2n} \\
\left(1+\frac{4}{n}\right)^{-1} &= \left(1-\frac{4}{n}+O(n^{-2})\right) = \left(1-\frac{4}{n}\right)\left(1+O(n^{-2})\right) \\
\left(1+\frac{4}{n}\right)^{2n} &= \exp\left[2n\left(\ln(1+\frac{4}{n})\right)\right] = \exp\left[2n\left(\frac{4}{n}-\frac{16}{2n^2}+O(n^{-3})\right)\right] = \\
&= \exp\left[8-\frac{16}{n}+O(n^{-2})\right] = e^8\left[1-\frac{16}{n}+O(n^{-2})\right] \\
\left(1+\frac{1}{n}\right)^{1-2n} &= \left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^{-2n}
\end{aligned}$$

$$\begin{aligned}
\left(1+\frac{1}{n}\right)^{-2n} &= \exp\left[-2n\left(\ln(1+\frac{1}{n})\right)\right] = \exp\left[-2n\left(\frac{1}{n}-\frac{1}{2n^2}+O(n^{-3})\right)\right] = \\
&= \exp\left[-2+\frac{1}{n}+O(n^{-2})\right] = e^{-2}\left(1+\frac{1}{n}+O(n^{-2})\right)
\end{aligned}$$

$$\left(1+\frac{3}{n+1}\right)^{2n-1} = e^6\left(1-\frac{18}{n}+O(n^{-2})\right)$$

$\boxed{\text{A7}}$ $O(n^{-5})$

$$\sum_{k=0}^{n} \frac{1}{n^2+k^2}$$

$$\begin{aligned}
\sum_{k=0}^{n} \frac{1}{n^2+k^2} &= \sum_{0\le k<n} \frac{1}{n^2+k^2} + \frac{1}{2n^2} = \int_0^n \frac{dx}{n^2+x^2} + \frac{1}{2n^2} + B_1 \frac{1}{n^2+x^2}\Big|_0^n + \frac{B_2}{2!}\frac{-2x}{(n^2+x^2)^2}\Big|_0^n + \\
&+ \quad O(n^{-5}) = \frac{\pi}{4n} + \frac{3}{4n^2} - \frac{1}{24n^2} + O(n^{-5})
\end{aligned}$$

$\boxed{\text{A8}}$ $O(n^{-5})$

$$\sum_{k=0}^{n} \frac{1}{n^2+2k^2}$$

$$\sum_{k=0}^{n}\frac{1}{n^2+2k^2} = \sum_{0\le k<n}\frac{1}{n^2+2k^2}+\frac{1}{3n^2}=\int_0^n\frac{dx}{n^2+2x^2}+\frac{1}{3n^2}+B_1\frac{1}{n^2+2x^2}\Big|_0^n+\frac{B_2}{2!}\frac{-4x}{(n^2+2x^2)^2}\Big|_0^n+$$
$$+\quad O(n^{-5})=\frac{1}{n\sqrt{2}}\cdot\operatorname{arctg}\sqrt{2}+\frac{2}{3n^2}-\frac{1}{27n^3}+O(n^{-5})$$

$\boxed{\text{A9}}$ $O(n^{-5})$

$$\sum_{k=0}^{n}\frac{1}{n^2+2k}$$

$$\sum_{k=0}^{n}\frac{1}{n^2+2k} = \sum_{0\le k<n}\frac{1}{n^2+2k}+\frac{1}{n^2+2n}=\int_0^n\frac{dx}{n^2+2x}+\frac{1}{n^2+2n}+B_1\frac{1}{n^2+2x}\Big|_0^n+\frac{B_2}{2!}\frac{-2}{(n^2+2x)^2}\Big|_0^n+$$
$$+\quad O(n^{-6})=$$

$\boxed{\text{A10}}$ $O(n^{-2})$

$$e^{H_{n+1}}$$

$$\begin{aligned}
e^{H_{n+1}} &= e^{H_n+\frac{1}{n+1}}\\
H_n &= \ln n+\gamma+\frac{1}{2n}-\frac{1}{12n^2}+O(n^{-4});\quad \frac{1}{n+1}=\frac{1}{n}-\frac{1}{n^2}+O(n^{-3})\\
e^{H_{n+1}} &= e^{\ln n+\gamma+\frac{1}{2n}-\frac{1}{12n^2}+O(n^{-4})+\frac{1}{n}-\frac{1}{n^2}+O(n^{-3})}=n\cdot e^{\gamma}\cdot e^{\frac{3}{2n}}\cdot e^{\frac{-13}{12n^2}}\cdot e^{O(n^{-3})}\\
&= n\cdot e^{\gamma}\cdot\left[1+\frac{3}{2n}+\frac{9}{8n^2}+O(n^{-3}\right]\cdot\left[1-\frac{13}{12n^2}+O(n^{-4})\right]\cdot\left[1+O(n^{-3})\right]\\
&= e^{\gamma}\cdot\left[n+\frac{3}{2}-\frac{1}{24n}+O(n^{-2})\right]
\end{aligned}$$

$\boxed{\text{A11}}$ Odhadnite s presnosťou $O(n^{-3})$

$$\begin{aligned}
S_n &= \sum_{0\le k\le n}\frac{1}{n^2+2k+1}=\frac{1}{n^2}\cdot\sum_{0\le k\le n}\frac{1}{1+\frac{2k+1}{n^2}}=\frac{1}{n^2}\sum_{0\le k\le n}\left[1-\frac{2k+1}{n^2}+O(n^{-2})\right]=\\
&= \frac{n+1}{n^2}-\frac{(n+1)^2}{n^4}+O(n^{-3})=\frac{1}{n}+O(n^{-3}).
\end{aligned}$$

$\boxed{\text{A12}}$ Odhadnite s presnosťou $O(n^{-3})$

$$\begin{aligned}
(n-2)! &= \frac{n!}{n(n-1)}=\frac{\sqrt{2\pi n}}{n(n-1)}\left(\frac{n}{e}\right)^n\left(1+\frac{1}{12n}+\frac{1}{288n^2}+O(n^{-3})\right)\\
\frac{1}{n(n-1)} &= \frac{1}{n^2}\cdot\left(1+\frac{1}{n}+\frac{1}{n^2}+O(n^{-3})\right).\\
(n-2)! &= \frac{\sqrt{2\pi n}}{n^2}\left(\frac{n}{e}\right)^n\left(1+\frac{13}{12n}+\frac{313}{288n^2}+O(n^{-3})\right)
\end{aligned}$$

$\boxed{\text{A13}}$ Odhadnite s presnosťou $O(n^{-3})$

$$\left(\frac{n+a}{n+b}\right)^{n+c},$$

kde $a, b, c \in Z$.

10 Nezaradené

generujuce funkcie pre Fibonacciho cisla

$$
\begin{aligned}
F_n &\Leftrightarrow \frac{z}{1-z-z^2} && (1)\\
F_{2n} &\Leftrightarrow \frac{z}{1-3z+z^2} && (2)\\
F_{3n} &\Leftrightarrow \frac{2z}{1-4z-z^2} && (3)\\
F_{4n} &\Leftrightarrow \frac{3z}{1-7z+z^2} && (4)\\
F_{5n} &\Leftrightarrow \frac{5z}{1-11z-z^2} && (5)\\
F_{6n} &\Leftrightarrow \frac{8z}{1-18z+z^2} && (6)\\
F_{7n} &\Leftrightarrow \frac{13z}{1-29z-z^2} && (7)\\
F_{8n} &\Leftrightarrow \frac{21z}{1-47z+z^2} && (8)\\
F_{9n} &\Leftrightarrow \frac{34z}{1-76z-z^2} && (9)\\
F_{10n} &\Leftrightarrow \frac{55z}{1-123*z+z^2} && (10)\\
F_{11n} &\Leftrightarrow \frac{89z}{1-199z-z^2} && (11)\\
F_{12n} &\Leftrightarrow \frac{144z}{1-322z+z^2} && (12)\\
F_{mn} &\Leftrightarrow \frac{F_m \cdot z}{1-a_n \cdot z+(-1)^m \cdot z^2} && (13)
\end{aligned}
$$

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