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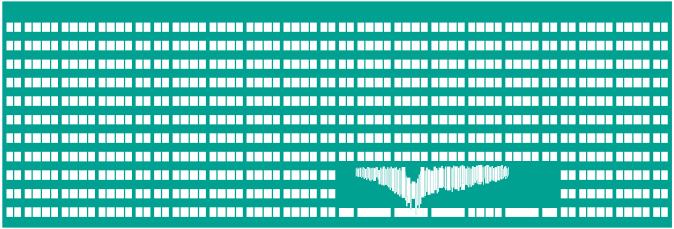

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# Nice Application of Complete Graph Decomposition

Petr Kovář

joint work with

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VSB – Technical University of Ostrava, Czech Republic

Seminár z teórie grafov,
11.3. 2021, Bratislava

# Outline of the talk

# Motivation

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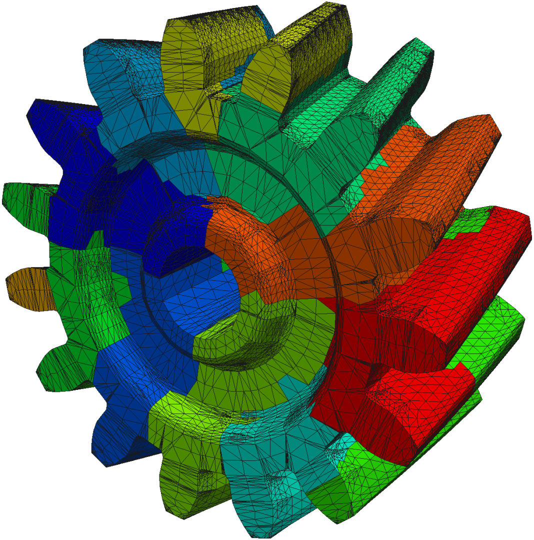
We have a dense matrix!

- distribute $N$ blocks to each processor
- (geometrically) closely related blocks to the same processor

# FEM & BEM



Many points, split into N machines.

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Parallel machine without shared memory.

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To each CPU as few different indices as possible.
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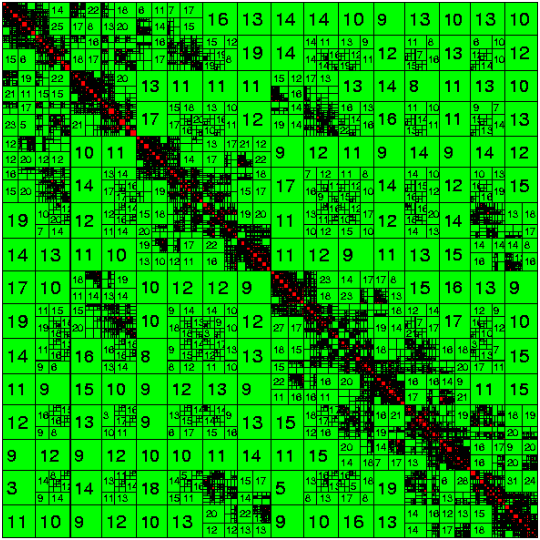
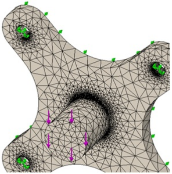
Not all $N^2$ blocks fit into the memory of one CPU!
(nor all $n^2$ elements of the matrix)

# Block matrix with numbers related to difficulty



Question: which $N$ blocks on which CPU?

- one diagonal block (longest computation)
- $(N-1)$ non-diagonal blocks



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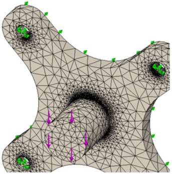

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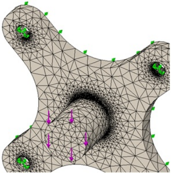

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- decomposing $K_N$ into $N$ subgraphs $G_1, G_2, \ldots, G_N$
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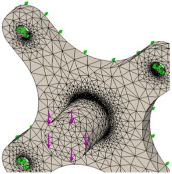

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First suppose $G_1, G_2, \ldots, G_N$ isomorphic to complete graph.

# Graph theory formulation

Decompose $K_N$ into $N$ copies of $K_k$ – dense subgraphs.
Necessary condition: $N = k^2 - k + 1$.

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Necessary condition: $N = k^2 - k + 1$.

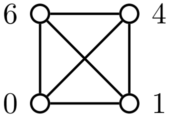
Particularly easy if the decomposition is cyclic:

## Definition (Graceful labeling)

Let $G$ be a graph with $m$ edges and a vertex labeling $\lambda : V(G) \to \{0, 1, \ldots, m\}$. The *length* of an edge $xy$ is

$$\ell(x, y) = \min\{|\lambda(x) - \lambda(y)|,\ 2m + 1 - |\lambda(x) - \lambda(y)|\}.$$

We call $f$ a *graceful* labeling if the set of edge lengths $\{\ell(x, y) : xy \in E(G)\} = \{1, 2, \ldots, m\}$.



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**Definition ($\rho$-labeling)**

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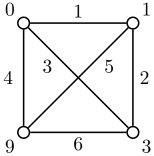
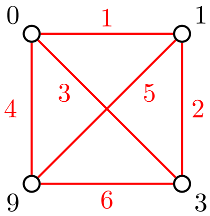
We say $\lambda$ is a *$\rho$-labeling* labeling if the set of edge lengths $\{\ell(x, y) : xy \in E(G)\} = \{1, 2, \dots, m\}$.



# $\rho$-labeling

**Theorem (Rosa 1967)**

*A graph $G$ with $m$ edges allows a cyclic decomposition of $K_{2m+1}$ iff $G$ has a $\rho$-labeling.*



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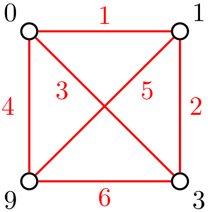
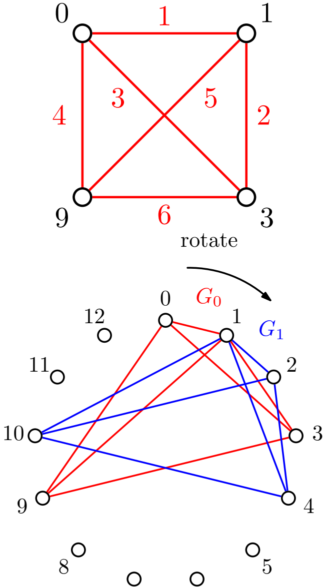



- Not all complete graphs have a $\rho$-labeling,
- infinitely many complete graphs have.

# Example



# Equivalent formulation

## Definition (perfect difference sets)

A set of integers $\{a_1, a_2, \ldots, a_k\} \subseteq [0, N]$ such that every nonzero residue modulo $N$ can be uniquely expressed in the form $a_i - a_j$.

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## Example

$\{0, 1, 4, 6\}$ is a perfect difference set for $m = 6$:
$1 = 1 - 0$, $2 = 6 - 4$, $3 = 3 - 1$, $4 = 4 - 0$, $5 = 6 - 1$, $6 = 6 - 0$.

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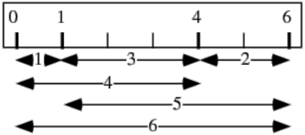
*Perfect ruler* (Guy 1994) has $k$ distinct marks s.t. any distance $1, 2, 3, 4, \ldots, N$ can be measured. E.g. 0, 1 ,4, 6



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Based on the perfect difference set $\{0, 1, 4, 6\}$ for $m = 6$ we can decompose $K_{2m+1} = K_{13}$ cyclically.

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A 13 by 13 block matrix to 13 machines, so that:

- 1 computation of diagonal block + 12 non-diagonal blocks to each process
- 4 rows (and 4 columns) of the geometry to each process

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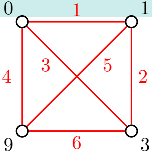
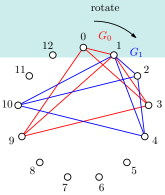
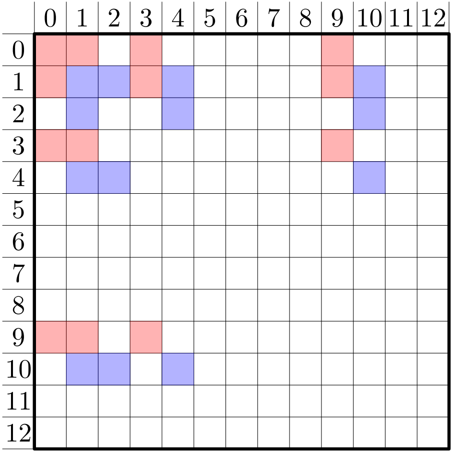
- the higher $N$ ($N = 13$) the better ratio

# Example


0
1
9
3
1
2
3
4
5
6


rotate
$G_0$
$G_1$
0
1
2
3
4
5
6
7
8
9
10
11
12


0 1 2 3 4 5 6 7 8 9 10 11 12
0
1
2
3
4
5
6
7
8
9
10
11
12

# Yet another formulation

Decomposing a complete graph into complete subgraphs – this seems to be a design theory problem.

## Definition

A *block design (BIBD)* is a collection $B$ of $b$ subsets (called blocks) of a finite set $X$ of $v$ elements such that any element of $X$ is contained in the same number $r$ of blocks, every block has the same number $k$ of elements, and each pair of distinct elements appear together in the same number $\lambda$ of blocks.
A symmetric BIBDs are (also known as 2-designs) are denoted as $2-(v,k,\lambda)$ designs has $b=v$. ($b \geq v$ by Fisher's inequality.)

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In our case:

- $k = k$
- $v = N = k^2 - k + 1$
- $b = N$ (symmetric)
- $\lambda = 1$

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Overview for $k < 100$ on web (Baumert):
`http://www.ccrwest.org/diffsets/diff_sets/baumert.html`

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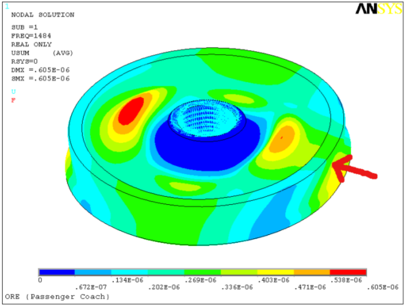
| n | N=31, k=6 | N=91, k=10 | N=133,k=12 |
|---|---|---|---|
| 12 288 | 175 MB, 1 s | 200 MB, 1 s | 207 MB, 1 s |
| 196 608 | 353 MB, 53 s | 280 MB, 25 s | 276 MB, 18 s |
| 786 432 | 999 MB, 294 s | 570 MB, 110 s | 535 MB, 99 s |
| 3 145 728 | | | 1911 MB, 596 s |

Format: average memory [MB], CPU time per process [s]

Motivation: railway wheel noise elimination by profiling




Courtesy of J. Szweda, Department of Mechanics, VŠB – TUO

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Remove 5 rows and columns:

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Computational time depends on the largest graph:
preferably few small and a many large dense graphs (complete or "almost" complete graphs).

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**Decomposition needs not to be cyclic**

$K_7 - K_{3,3}$ decomposes $K_{25}$, yet no $\rho$-labeling.
reference?

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*Let $r, s$ be odd. If $G$ decomposes $K_r$ into $r$ copies and $H$ decomposes $G[\overline{K_s}]$ into $s$ copies, then a dense graph $X$ on $|H|$ vertices decomposes $K_{rs}$ into $rs$ copies of $X$.*

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## Theorem

*If $r = p^2 - p + 1$ and $s = q^2 - q + 1$, then we can decompose $K_{rs}$ into $rs$ dense (for small $s$) isomorphic subgraphs on $pq$ vertices.*

## Example

Decompose $K_{147}$ $(147 = 7 \cdot 21)$ into 147 isomorphic subgraphs on 15 vertices.

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If $q$ is even, we decompose $K_{pq}$ into $pq$ subgraphs $H_1, H_2, \ldots, H_{pq}$ (not isomorphic).

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If $q$ is even, we decompose $K_{pq}$ into $pq$ subgraphs $H_1, H_2, \ldots, H_{pq}$ (not isomorphic).

**Lemma**

*Let $p$ be odd and $q$ even. If $G$ decomposes $K_p$ into $p$ copies and $H$ decomposes $G[\overline{K_q}]$ into $q$ copies, then $K_{pq}$ can be decomposed into $pq$ dense subgraphs $X_1, X_2, \ldots, X_{pq}$ each on $|H|$ vertices.*

Leads to a recursive construction.
Isolated values (case by case).

# Further approximations

Up to $N = 1000$ ($k \simeq 40$):

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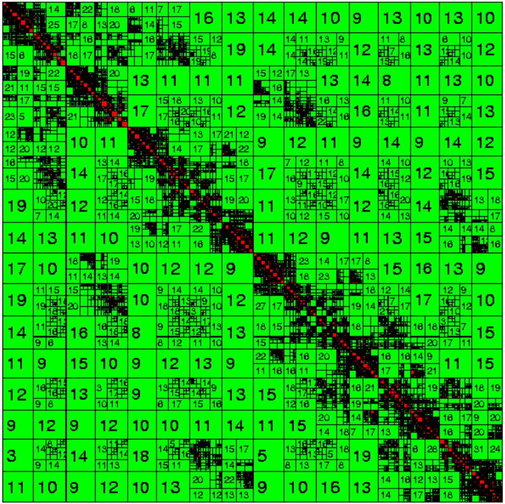
Theoretical optimum: 25 vertices.

# Optimization?



. . . balance the sums among the subgraphs.

Thank you for your attention.