The Effect of Local Symmetry-preserving Operations on the Connectivity of Embedded Graphs

Heidi Van den Camp

Department of Applied Mathematics, Computer science and Statistics

February 25, 2021

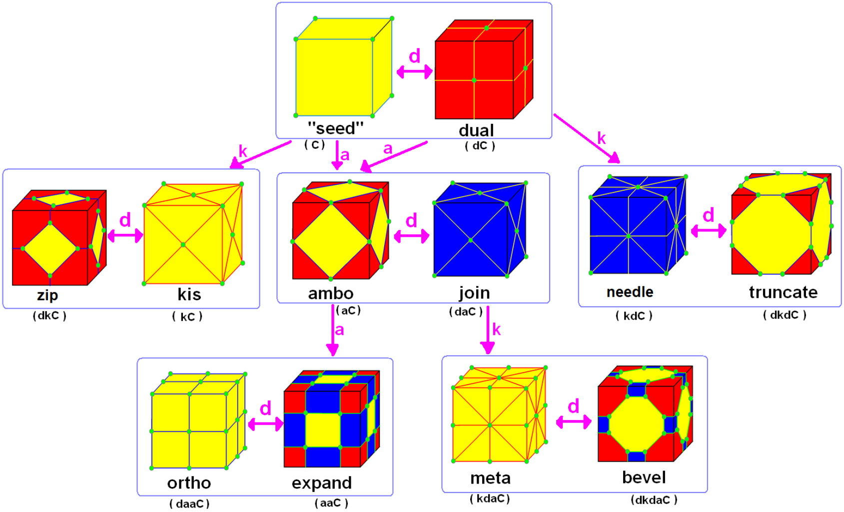
GHENT UNIVERSITY

Overview

Overview

Introduction

Local symmetry-preserving operations



Introduction

Local symmetry-preserving operations

- Polyhedra and operations on them have been studied for a long time
- No uniform way of describing all symmetry-preserving operations
- Lsp-operations defined in 2017[1]

---

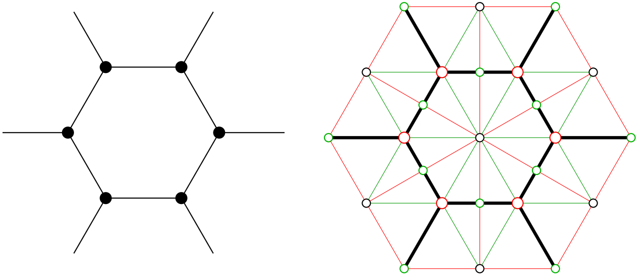
[1] Gunnar Brinkmann, Pieter Goetschalckx, and Stan Schein. "Comparing the constructions of Goldberg, Fuller, Caspar, Klug and Coxeter, and a general approach to local symmetry-preserving operations". In: *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* 473.2206 (2017), p. 20170267.

Introduction

Local symmetry-preserving operations

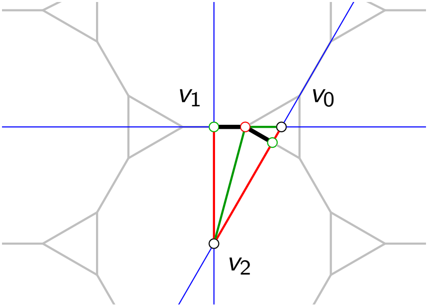
Definition

The *barycentric subdivision* B_G of an embedded graph G has a vertex for every vertex, edge and face of G. We say that these vertices are of type *type 0, 1 and 2* respectively. The adjacency is shown in the drawing.



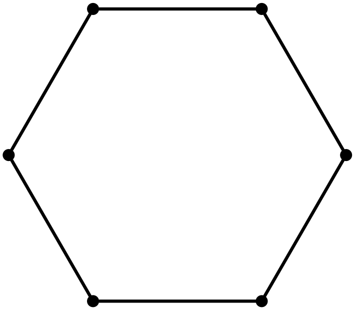
An *lsp-operation* is a triangle, originating from the barycentric subdivision of a tiling T of the plane, such that:

- T is 3-connected
- the triangle has angles of 30°, 60° and 90°
- the edges of the triangle are on symmetry axes of T



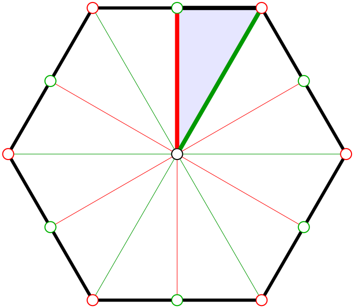
Introduction

Local symmetry-preserving operations



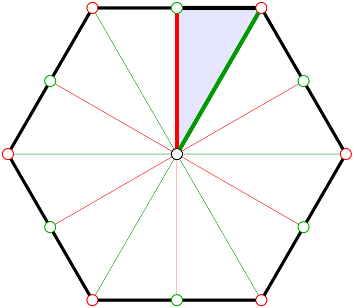
Introduction

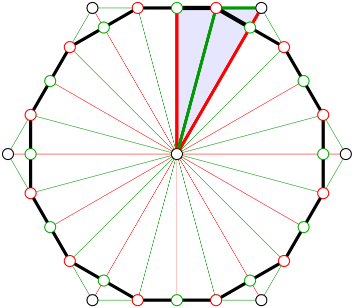
Local symmetry-preserving operations



Introduction

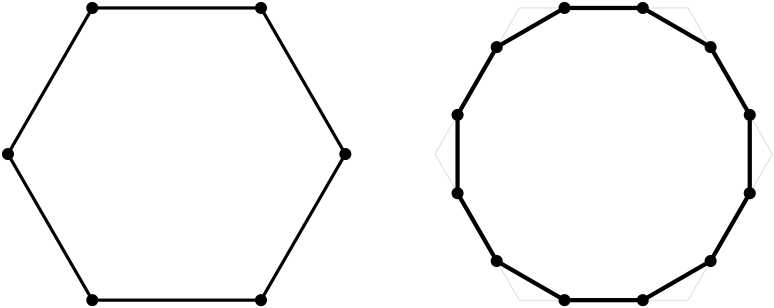
Local symmetry-preserving operations





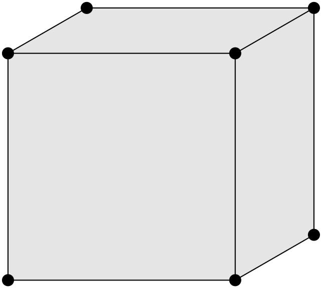
Introduction

Local symmetry-preserving operations



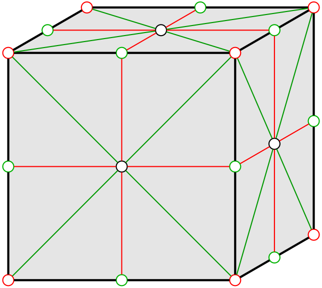
Introduction

Local symmetry-preserving operations



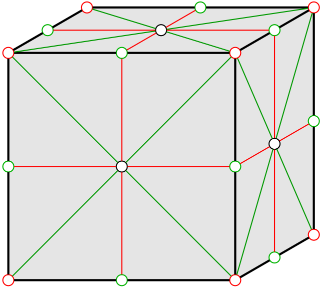
Introduction

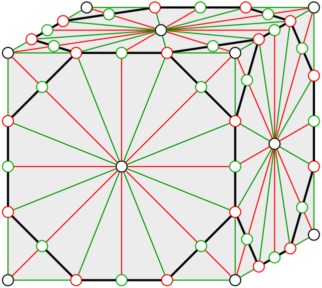
Local symmetry-preserving operations



Introduction

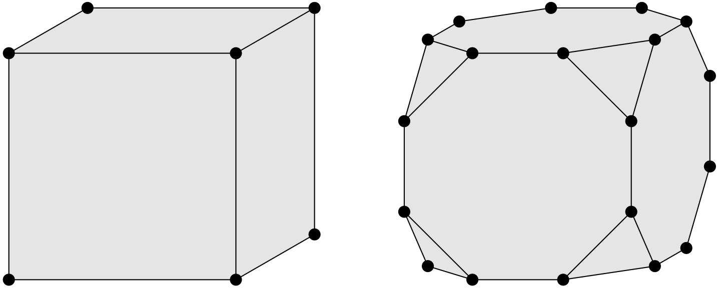
Local symmetry-preserving operations





Introduction

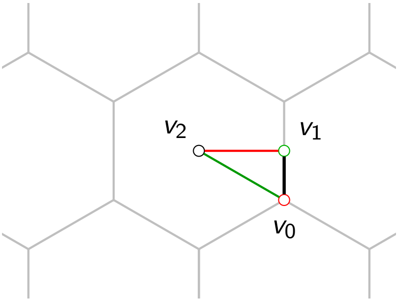
Local symmetry-preserving operations

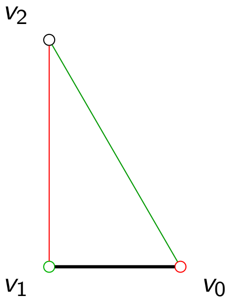


Introduction

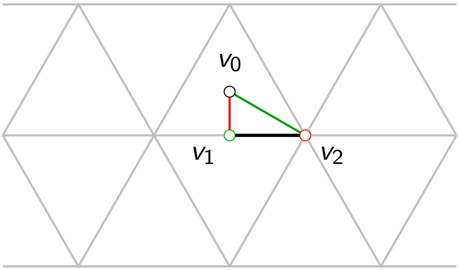
Local symmetry-preserving operations

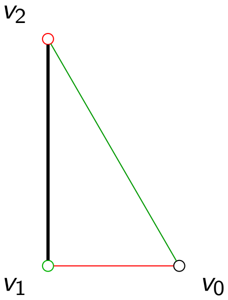
The identity



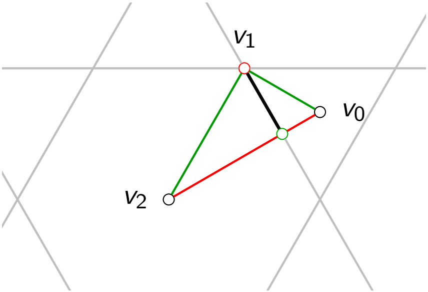


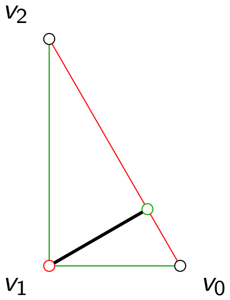
The dual



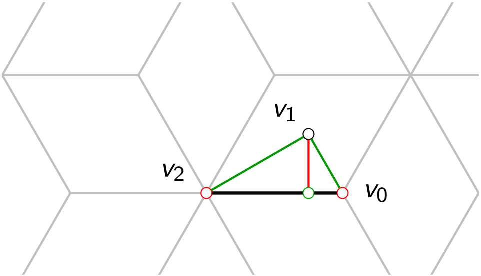


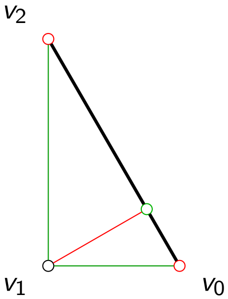
Ambo





Join





Theorem

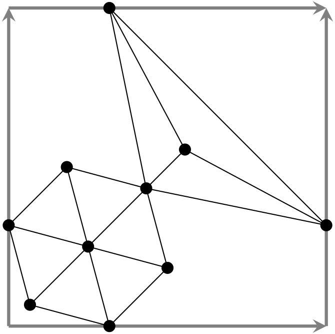
The result of applying an lsp-operation to a 3-connected plane graph is a 3-connected plane graph.[2]

- Extend to all embedded graphs?

[2] Gunnar Brinkmann, Pieter Goetschalckx, and Stan Schein. "Comparing the constructions of Goldberg, Fuller, Caspar, Klug and Coxeter, and a general approach to local symmetry-preserving operations". In: *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* 473.2206 (2017), p. 20170267.

Theorem

There exist k-connected embedded graphs with a simple dual that has a 1-cut.[3]



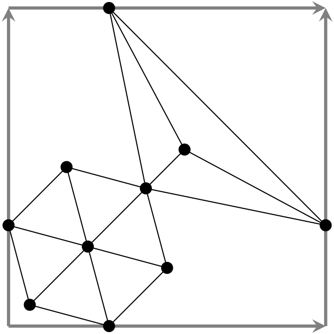
[3] Drago Bokal, Gunnar Brinkmann, and Carol T Zamfirescu. "The Connectivity of the Dual". In: *arXiv preprint arXiv:1812.08510* (2018).

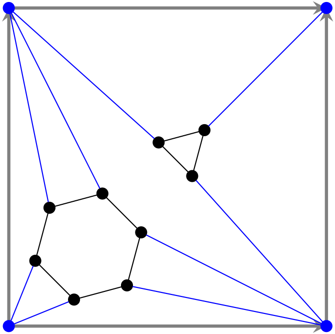
Introduction

Known results

Theorem

There exist k-connected embedded graphs with a simple dual that has a 1-cut.[3]





[3] Drago Bokal, Gunnar Brinkmann, and Carol T Zamfirescu. "The Connectivity of the Dual". In: *arXiv preprint arXiv:1812.08510* (2018).

Theorem

Let G be a 3-connected embedded graph such that $fw(G) \geq 3$. Then G^* is also 3-connected, and $fw(G^*) \geq 3$.[4]

- In general, 3-connectivity is not preserved for higher genus
- Maybe it is for embeddings of face-width at least 3?

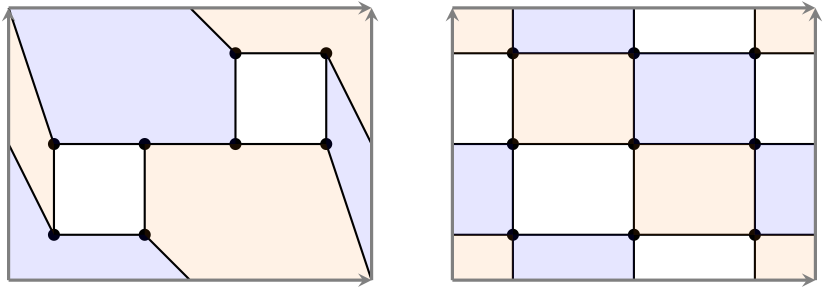
[4] Bojan Mohar. "Face-width of embedded graphs". In: *Mathematica Slovaca* 47.1 (1997), pp. 35–63.

Overview

Face-width at least 3

Definition

Let G be an embedded graph. The *face-width* $fw(G)$ of G is the minimal number of faces of G for which their union can contain a non-contractible curve.



Face-width at least 3

Theorem

Let G be a 3-connected embedded graph such that $fw(G) \geq 3$, and let O be an lsp-operation. Then $O(G)$ is also 3-connected and $fw(O(G)) \geq 3$.

This implies:

- Dual(G) is 3-connected
- Ambo(G) is 3-connected
- Truncate(G) is 3-connected
- Join(G) is 3-connected
- ...

Overview

Theorem

There exist 3-connected embedded graphs with a simple dual that is not 3-connected.[5]

- Intuitively, the dual is the 'worst' case
- Are there other lsp-operations that do not preserve 3-connectivity?

[5] Drago Bokal, Gunnar Brinkmann, and Carol T Zamfirescu. "The Connectivity of the Dual". In: *arXiv preprint arXiv:1812.08510* (2018).

General embedded graphs

Theorem

Given an lsp-operation O, the embedded graph $O(G)$ is 3-connected for any 3-connected embedded graph G if and only if O is edge-preserving.

General embedded graphs

Idea of the proof

- Want to use 3-connectivity of G to prove 3-connectivity of $O(G)$
- What happens to vertices and edges of G in $O(G)$?

General embedded graphs

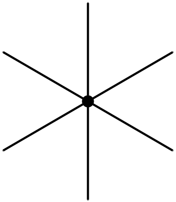
Idea of the proof

- Assume that G is 3-connected and $O(G)$ is not 3-connected.
- There is a set S of two vertices in $O(G)$ such that $O(G) \setminus S$ is not connected.
- Let x and y be two vertices in different components of $O(G) \setminus S$
- Associate two vertices x_G and y_G of G with x and y
- Use 3-connectivity of G to find a path from x_G to y_G in G that induces a path from x to y in $O(G) \setminus S$

General embedded graphs

Idea of the proof

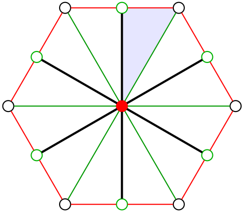
Vertices of G in $O(G)$:



General embedded graphs

Idea of the proof

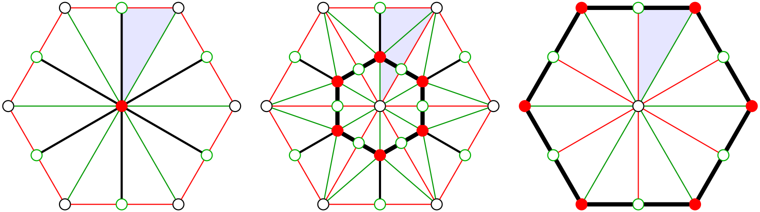
Vertices of G in $O(G)$:



General embedded graphs

Idea of the proof

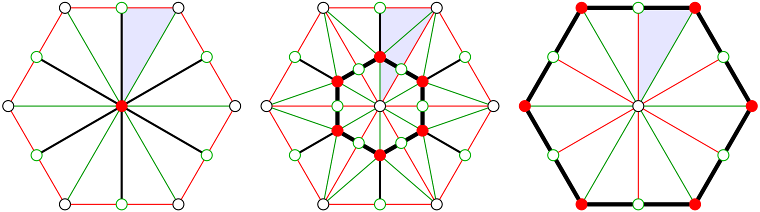
Vertices of G in $O(G)$:



General embedded graphs

Idea of the proof

Vertices of G in $O(G)$:



Theorem

The dual is the only lsp-operation for which two vertices of $O(G)$ can 'break' more than one vertex of G.

General embedded graphs

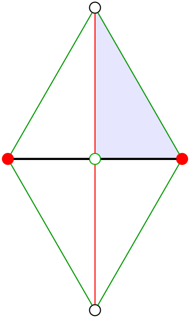
Idea of the proof

Edges of G in $O(G)$

General embedded graphs

Idea of the proof

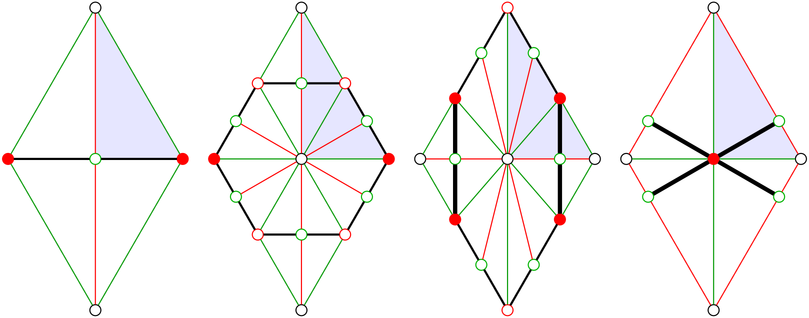
Edges of G in $O(G)$



General embedded graphs

Idea of the proof

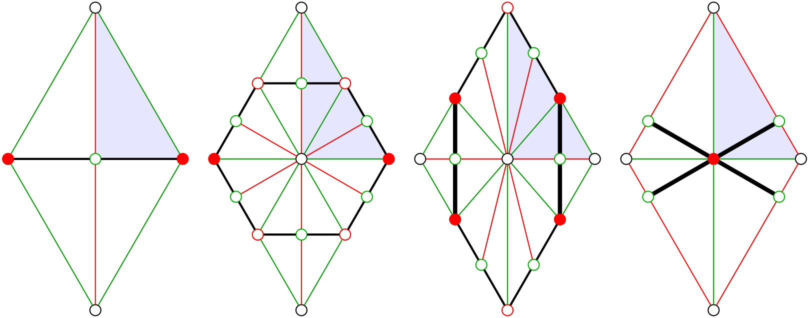
Edges of G in $O(G)$



General embedded graphs

Idea of the proof

Edges of G in $O(G)$



Theorem

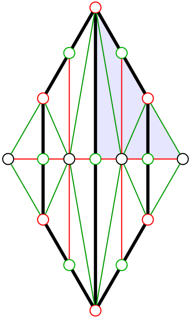
If O is an edge-preserving operation, then a set of two vertices in $O(G)$ can 'break' at most two edges of G.

General embedded graphs

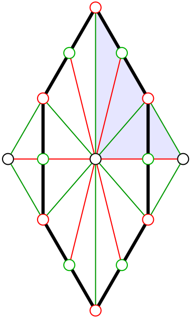
Idea of the proof

Definition

An lsp-operation is *edge-breaking* if v_2 is of type 0, and it is adjacent to v_1 in O. An lsp-operation that is not edge-breaking is *edge-preserving*.



type 1



type 2

- Assume that G is 3-connected and $O(G)$ is not 3-connected.
- There is a set S of two vertices in $O(G)$ such that $O(G) \setminus S$ is not connected.
- There exist two vertices x and y in the vertex-images of two vertices a and b of G that are in different components of $O(G) \setminus S$
- S 'breaks' at most two vertices or edges of G
- As G is 3-connected, there is a path from a to b in G without these two vertices or edges
- This path induces a path from x to y in $O(G) \setminus S$

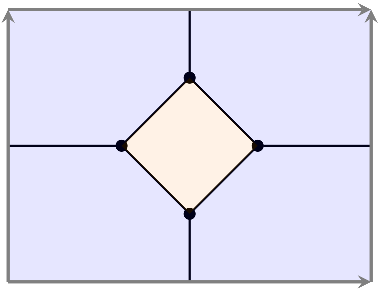
Theorem

Given an lsp-operation O, the embedded graph $O(G)$ is 3-connected for any 3-connected embedded graph G if and only if O is edge-preserving.

- Examples needed to prove the other implication

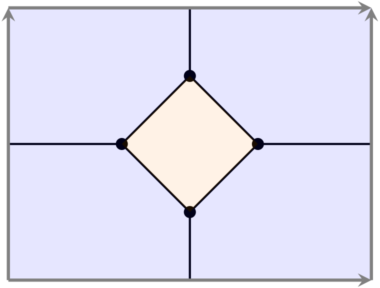
General embedded graphs

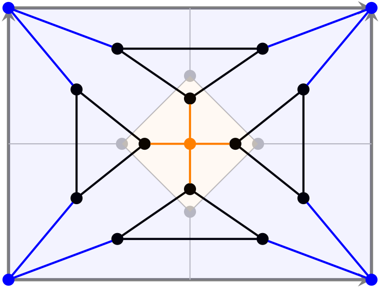
Examples



General embedded graphs

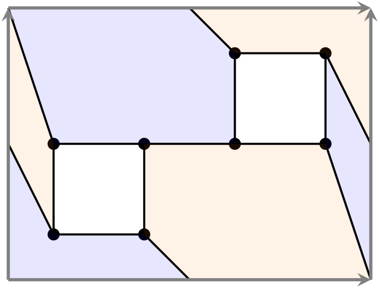
Examples





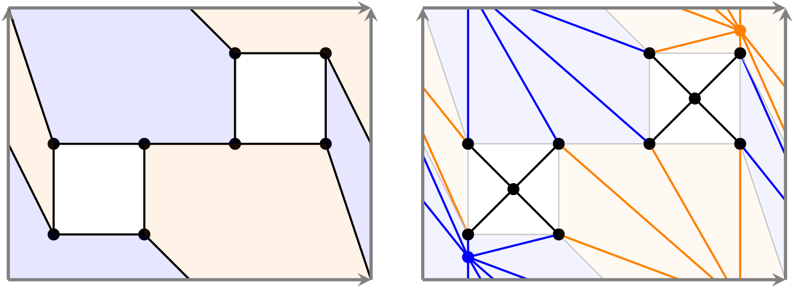
General embedded graphs

Examples



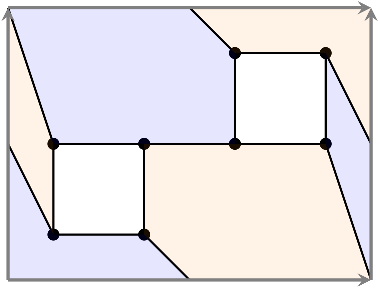
General embedded graphs

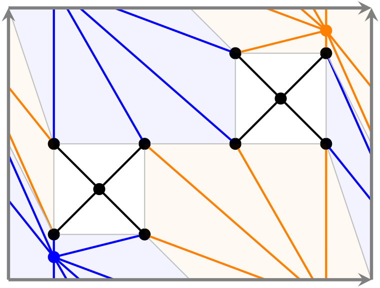
Examples



General embedded graphs

Examples

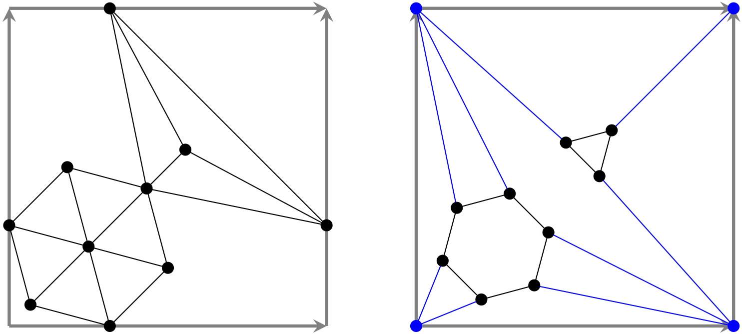




- None of these examples have a simple dual

General embedded graphs

Examples



Drago Bokal, Gunnar Brinkmann, and Carol T Zamfirescu. "The Connectivity of the Dual". In: *arXiv preprint arXiv:1812.08510* (2018)

General embedded graphs

Overview

Do these operations preserve 3-connectivity for all embedded graphs satisfying the given condition?

	Edge-breaking			Edge-preserving
	Dual	Type 2	Type 1′	
G plane	Yes	Yes	Yes	Yes
$fw(G) \geq 3$	Yes	Yes	Yes	Yes
G^* simple	No	Yes	Yes	Yes
$O(G)$ simple	No	No	Yes	Yes
General	No	No	No	Yes