Normal edge-colorings of cubic graphs

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Outline

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Definitions

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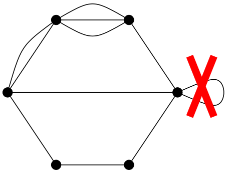

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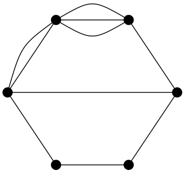

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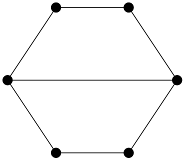

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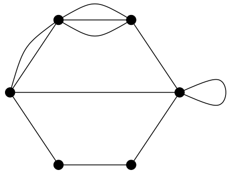
- A pseudograph admit both parallel edges and loops.



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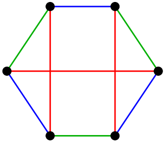
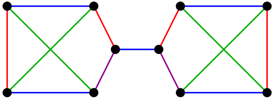
Edge-colorings

- A graph is *k-edge-colorable*, if its edges can be colored with k colors such that adjacent edges receive different colors.



3–edge–coloring



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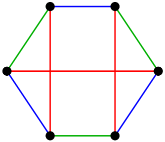
- A graph is *k*-edge-colorable, if its edges can be colored with k colors such that adjacent edges receive different colors.
- The least k, for which a graph G is k-edge-colorable, is called chromatic index of G and is denoted by $\chi'(G)$.



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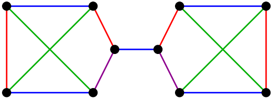

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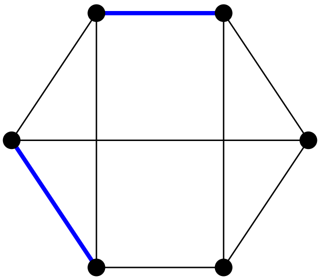
Independent edges and Matchings

- A matching is a set of edges of a graph such that any two edges are independent.



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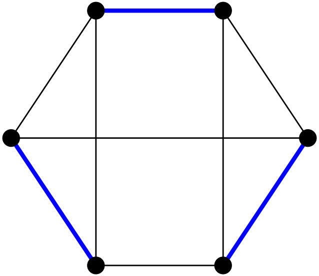
- A matching is perfect, if it contains $\frac{|V|}{2}$ edges.



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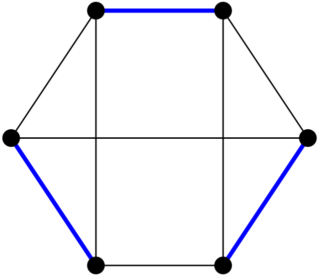
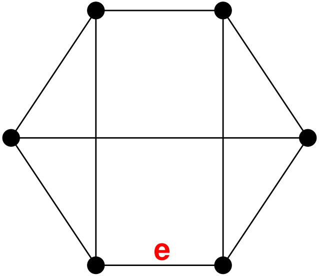


Cubic graphs and perfect matchings

Proposition

Any edge of a bridgeless cubic graph G belongs to a perfect matching of G.



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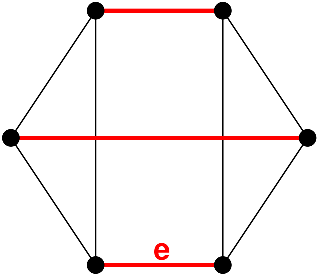

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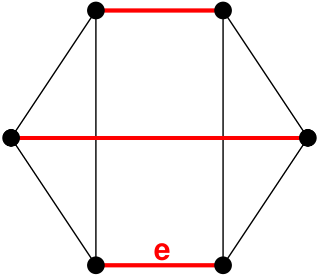
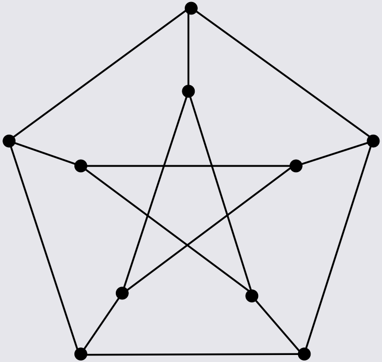


Definition

For a bridgeless cubic graph G, let $k(G)$ be the smallest number of perfect matchings covering the edge-set of G.

The Petersen graph P and its 6 perfect matchings

The Petersen graph



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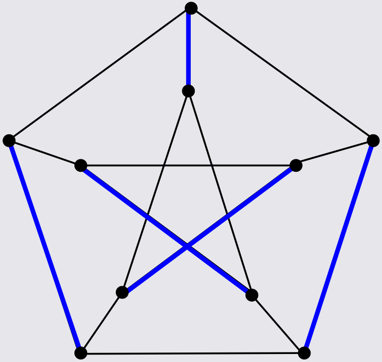

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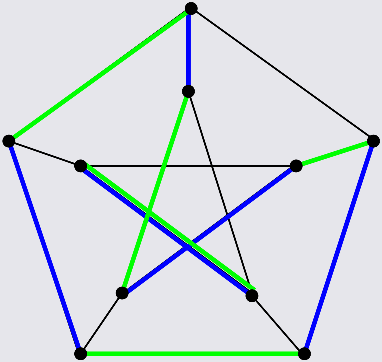

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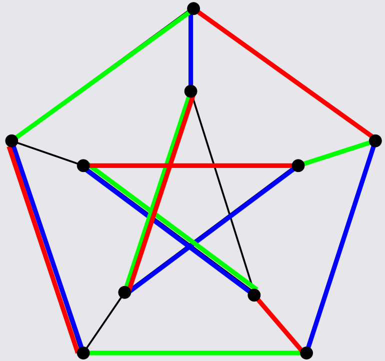

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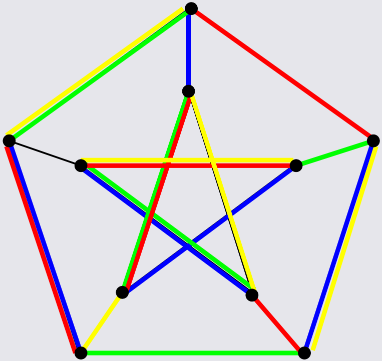

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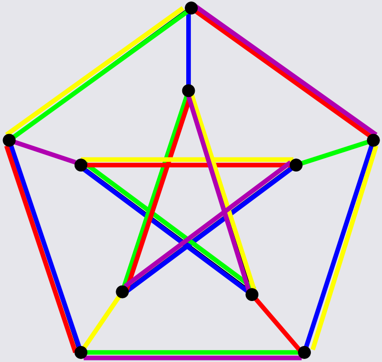

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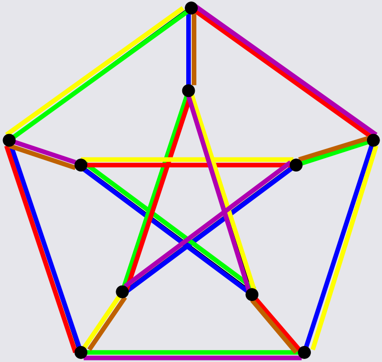

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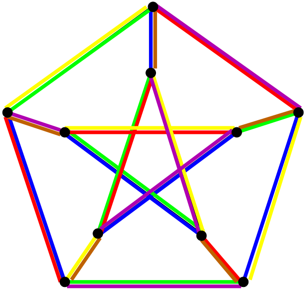

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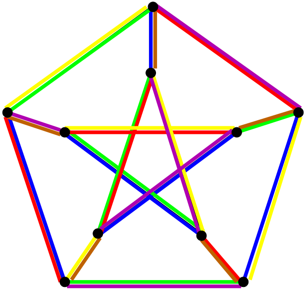

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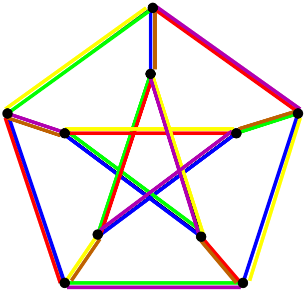

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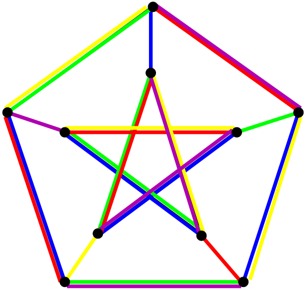

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Both trivial for 3*-edge-colorable cubic graphs.*

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Conjectures of Berge and Berge-Fulkerson are equivalent.

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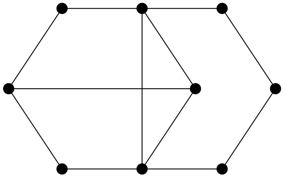

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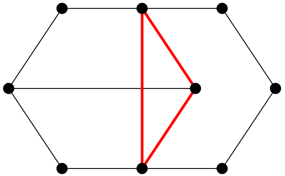

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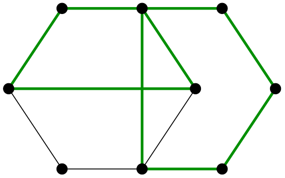

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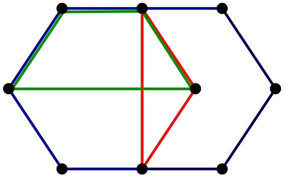

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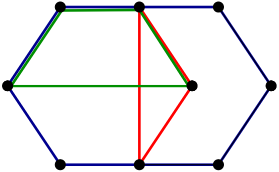

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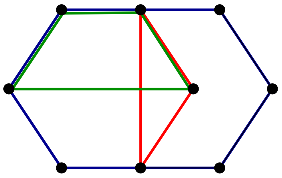

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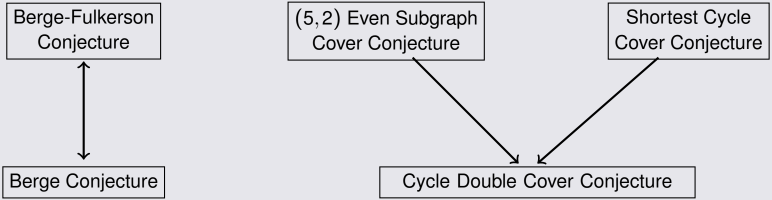

Figure 1: The relationship among the five conjectures.

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If G admits an H-coloring f, then we will write $H \prec G$.

An example of an H-coloring

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An example: $H \prec G$

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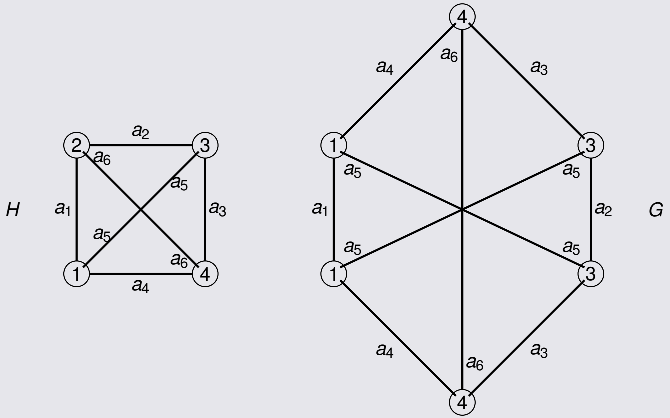

Figure 2: An example of an H-coloring of G.

Petersen coloring conjecture

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Every bridgeless cubic graph admits a Petersen coloring (i.e. $P \prec G$)

Consequences of Petersen Coloring conjecture

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Petersen Coloring conjecture implies Shortest Cycle Cover Conjecture.

The relationship among the six conjectures

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The relationship

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The relationship

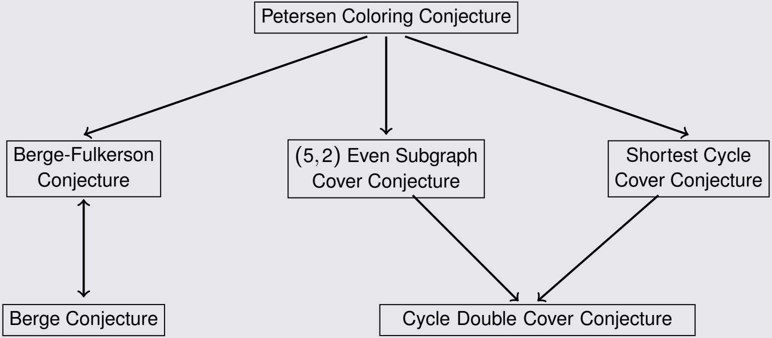


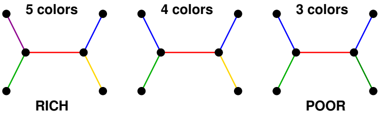
Figure 3: The relationship among the six conjectures.

Poor, rich edges and normal edge-colorings of cubic graphs

Definition

Let c be a k-edge-coloring of a cubic graph G, and let $S_c(w)$ be the set of colors of edges of G incident to the vertex w. Then an edge $e = uv$ is

- POOR, if $|S_c(u) \cup S_c(v)| = 3$,
- RICH, if $|S_c(u) \cup S_c(v)| = 5$.



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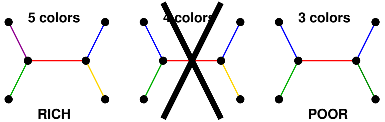

Definition

A k-edge-coloring c of a cubic graph G is NORMAL, if any edge of G is poor or rich in c. Denote by $\chi'_N(G)$ the smallest k for which G admits a normal k-edge-coloring (if it does exist).

An example of a normal edge-coloring of a cubic graph

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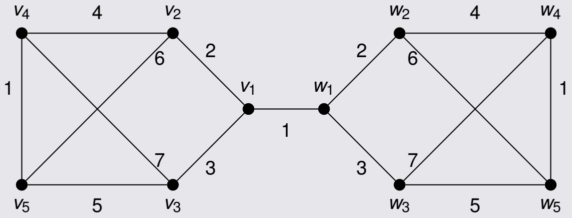

Figure 4: A cubic graph that requires 7 colors in a normal coloring.

An example of a normal edge-coloring of a cubic graph

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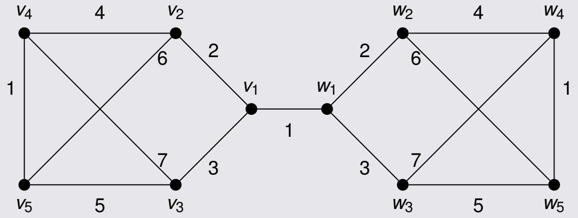

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The bridge is poor.

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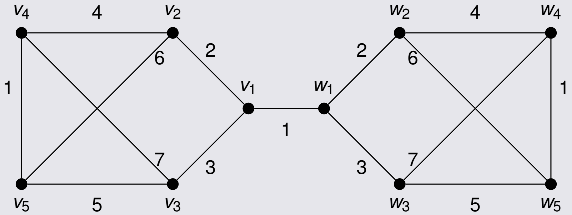

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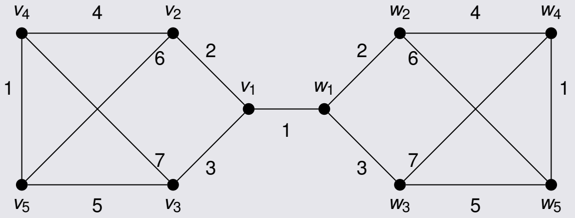

Figure 4: A cubic graph that requires 7 colors in a normal coloring.

The bridge is poor. All other edges are rich. It can be shown that $\chi'_N(G) = 7$.

An example of a cubic graph without a normal k-edge-coloring

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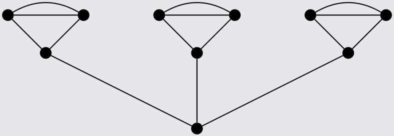

Figure 5: An example of a cubic graph that does not admit a normal coloring.

Why normal colorings are so important?

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Proposition (Jaeger, 1988)

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A cubic graph G admits a Petersen Coloring iff $\chi'_N(G) \leq 5$.

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For any bridgeless cubic graph G, we have $\chi'_N(G) \leq 5$.

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What is the smallest k, such that for any simple cubic graph G we have $\chi'_N(G) \leq k$*?*

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Nowhere-zero flows

The elementary Abelian group $\mathbb{Z}_2^3$

Let $\mathbb{Z}_2^3$ be the set of all binary triples. It is an abelian group of order 8 with respect to the sum, where the unit element is $(0, 0, 0)$.

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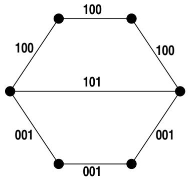

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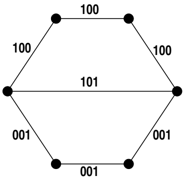


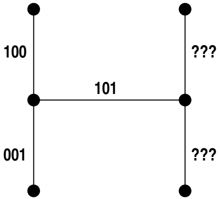
Theorem (Jaeger's 8-flow theorem)

Any bridgeless (not necessarily cubic) graph admits a nowhere-zero $\mathbb{Z}_2^3$-flow.

Proof idea of our main result: bridgeless cubic graphs

$\mathbb{Z}_2^3$-flows induces normal 7-edge-colorings in bridgeless cubic graphs

Let f be a nowhere-zero $\mathbb{Z}_2^3$-flow (guaranteed by Jaeger's 8-flow theorem). Clearly, it is a 7-edge-coloring. Let $e = uv$ be any edge of G. We have $e \in \partial_G(u) \cap \partial_G(v)$.



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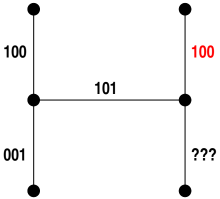

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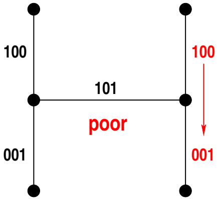

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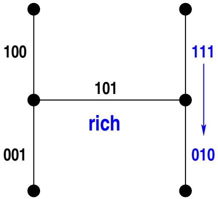
- If $\{f(e)\} = S_f(u) \cap S_f(v)$, then clearly e is rich in f.



Main ideas of our proof

1) We need to improve a bit some classical results on nowhere-zero flows by adding some LOCAL CONSTRAINS
2) Normal edge-colorings arising from 8-flows satisfy some ADDITIONAL PROPERTIES

Classical results on Nowhere-Zero Flows with LOCAL CONSTRAINS

Lemma (1)

Let G be a 4-edge-connected (pseudo)graph. Then G admits a nowhere-zero $\mathbb{Z}_2^2$*-flow*

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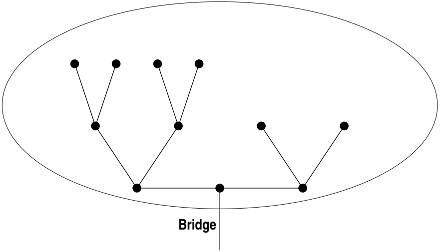
Normal edge-colorings arising from 8-flows satisfy some additional properties



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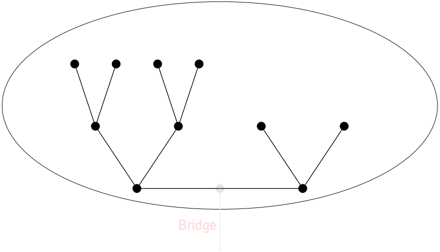

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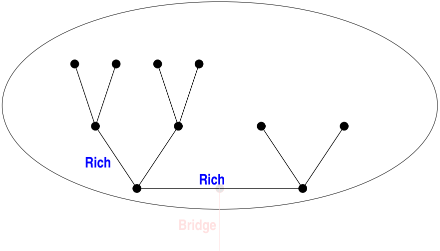

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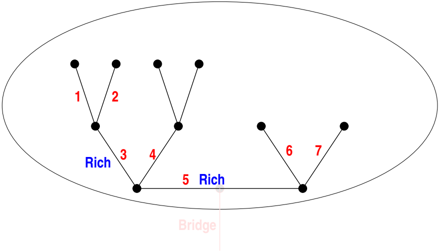

It could be that we use 7 different colors! There is NOT enough “space” to assign a color to the bridge in the original graph.

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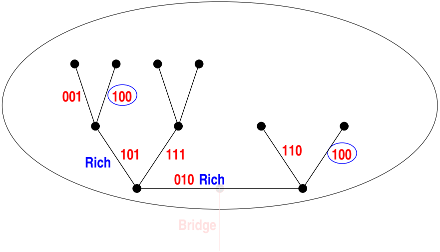

This is not the case by using 8-flow! Here, there is enough “space” to assign a color to the bridge in the original graph
(NOTE: it is not completely trivial).

Proof idea of our main result: simple cubic graphs with bridges

Step 1: subdivisions of bridgeless cubic graphs

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Next we show that if G' is a simple graph obtained from a bridgeless cubic graph G by subdividing one of its edges once and adding a pendant edge incident to the unique degree-two vertex, then $\chi'_N(G') \leq 7$.

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Step 2: arbitrary simple cubic graphs

Finally, we complete the proof of the main theorem by showing that for any simple cubic graph G we have $\chi'_N(G) \leq 7$.

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Conjecture (Intermediate conjecture)

For any bridgeless cubic graph G, we have $\chi'_N(G) \leq 6$.

A different approach

Question:

What is the minimum number of abnormal edges in a 5-edge-coloring of a bridgeless cubic graph?

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A related open problem:

Conjecture (Samal 2011)

Let G be a cubic bridgeless graph, M a perfect matching of G. Then, there is a 5-edge-coloring of G so that every edge not in M is either rich or poor.

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Proposition (Jin, Kang 2019)

Every bridgeless cubic graph G has a proper 5-edge-coloring such that at most $\frac{1}{5}|E(G)|$ edges are abnormal.

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Proposition V.Mkrtchyan, G.M. 2020

Showing a sublinear bound for the number of abnormal edges with respect to the order of a bridgeless cubic graph G is as hard as proving Petersen coloring conjecture.

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Moreover, we prove results of the following type:

Proposition V. Mkrtchyan, G.M. 2020

The following statements are equivalent:
(a) Any (cyclically) k-edge-connected cubic graph G admits a proper 5-coloring c, such that the number of abnormal edges is at most $2k+1$.
(b) There exists a sublinear function f, such that every (cyclically) k-edge-connected cubic graph G admits a proper 5-coloring c with at most $f(|V(G)|)$ abnormal edges.

THANKS FOR YOUR ATTENTION!